

On the Gaussian Interference Channel with Causal Cognition, or with Unilateral Source Cooperation

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Abstract

This paper considers the causal cognitive interference channel that consists of two full-duplex transmitter-receiver pairs sharing the same channel, where one transmitter can causally learn the message of the other transmitter through a noisy link. This channel models unilateral source cooperation. The work focuses on the generalized degrees-of-freedom of both the *symmetric* and *asymmetric* sum-capacity for the Gaussian noise channel. In the former case the interfering links and the direct links are of the same strength, in the latter case one of the interfering links is set to zero. It is shown through evaluation of various achievable schemes that known sum-rate upper-bounds are achievable to within a constant gap regardless of the strength of the channel parameters. The regimes where causal cognitive radio is equivalent, in terms of generalized degrees-of-freedom, to no-cooperation at all or to non-causal cooperation are identified. Interestingly, in all but one case the achievable schemes are quite simple in that only superposition coding is used – while it is shown that more complex schemes using binning could be used to achieve a smaller gap.

I. INTRODUCTION

A shorter version of this paper has been submitted to the 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton 2012).

We consider the Gaussian cognitive overlay paradigm shown in Fig. 1 consisting of two transmitters PTx and CTx and two receivers PRx and CRx. PTx and PRx are so-called *primary* nodes while CTx and CRx are *cognitive* nodes. The purposes of the techniques considered here are to firstly allow the cognitive nodes to communicate without hindering the communication of the primary nodes and secondly to enhance the communication reliability of the primary link. To this end, we exploit a lossy communication link between PTx and CTx and we assume that CTx can operate in full-duplex on the same communication channel (i.e. same carrier frequency). Moreover, we treat the case of causal transmission at CTx, in the sense that knowledge of the primary transmission is only used for encoding at CTx after some processing delay to allow for (partial) decoding or compression of the signal observed at CTx. We shall denote the above described system as Causal Cognitive Interference Channel (CCIC).

From an application standpoint, this model fits future 4G networks with relays [1] where CTx corresponds to the so-called *relay-node* (RN). In these networks, the RN communicates in a wireless fashion with PTx which is called the *Donor-eNB*. We consider deployment scenarios where the Donor-eNB→RN link operates on the same carrier frequency as the PTx→PRx and PTx→CRx links in full-duplex (Inband Relay type 1b [1]). The key difference in 4G network topologies is that the RN is constrained to operate in a decode-and-forward fashion. The second main difference is that in the PRx and CRx we are considering sophisticated interference-mitigation techniques which exploit knowledge of the codebooks used at both PTx and CTx, and moreover where the codebooks are conceived for the interference scenario (e.g. superposition-coding [2] or dirty-paper-coding [3]).

Different interference scenarios are considered and can correspond to the choice of appropriate deployment configurations in cognitive radio networks. The first class is the symmetric Gaussian channel where both interfering links and direct links are of the same strength. The second class is the interference-asymmetric Gaussian channel where either the link PTx→CRx is non-existent (*Z*-channel) or the link CTx→PRx is non-existent (*S*-channel). The *Z*-channel would model a situation such as an indoor CTx→CRx with another receiver (PRx) connected to an outdoor base station (PTx) in vicinity to CTx. The *S*-channel would model the case where PRx is out-of-range of CTx and the base station (PTx) schedules traffic to both PRx and CTx/CRx concurrently. Both scenarios are of practical relevance for cognitive radio deployment [1].

A. Related Work

The presence of a lossy communication link between PTx and CTx enables CTx to cooperate with PTx in order to send the PTx's message. CTx, in fact, through this noisy channel overhears the signal sent by the PTx and gathers information about

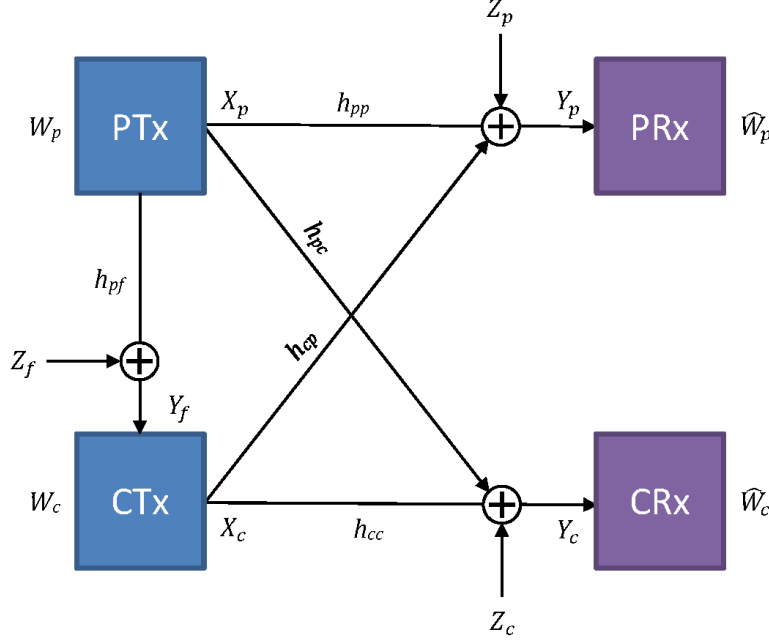


Fig. 1. The 2-user Gaussian Causal Cognitive Interference Channel.

the activity of PTx. This serves as a basis for unilateral cooperation between the two sources. Unilateral source cooperation is a special case of *generalized feedback* or bilateral cooperation [4].

Regarding bilateral source cooperation between sources, the largest known achievable region, to the best of our knowledge, is the one presented in [4]. Here each source splits the message into two sub-messages, i.e. *common* and *private*, as in the Han-Kobayashi's scheme for the non-cooperative channel [2]. Moreover each of these two messages is further sub-divided into a *non-cooperative* and a *cooperative* part. The former is transmitted as in the classical IC, the latter is delivered to the destination by exploiting the cooperation among the two sources. In [4] each source, e.g. source 1, after learning the message of source 2, cooperates by sending the common cooperative messages of source 2 and applies Dirty Paper Coding (DPC) [3] on the private cooperative part of source 2 riding its own receiver (receiver 1) of this interference.

Bilateral source cooperation has received lots of attention recently. Host-Madsen [5] first studied outer and inner-bounds for the sum-rate capacity for the Gaussian IC with either source or destination bilateral cooperation. Concerning the outer-bound, the author firstly evaluated the cut-set bound, which is easily computable, but turns out to be loose. Consequently, the author tightened the bound by extending to the cooperative case the bounds originally developed by Kramer [6] for the classical IC. Prabhakaran and Viswanath [7] extended the idea in [8, Th.1] and studied an outer-bound for the Gaussian IC with bilateral source cooperation, by treating the noises as independent (this assumption is not without loss of generality for the general cooperative IC). Tandon and Ulukus [9] developed an outer-bound for the IC with bilateral source cooperation based on dependence-balance idea of Hekstra and Willems [10] and proposed a novel method to evaluate it in the Gaussian channel with independent noises. Tuninetti [11] derived a general outer-bound for the IC with bilateral cooperation by extending Kramer's idea [6, Th.1] to any memoryless IC with source cooperation.

Concerning unilateral source cooperation between sources, in [12] the authors study both the case when the collaborating transmitter works in full-duplex and half-duplex mode. With regard to the full-duplex mode, they develop two achievable schemes: one exploits Partial-Decode-Forward relaying and Gelfand-Pinsker binning and the second extends the first by adding rate splitting and superposition coding. These two schemes can be obtained as special cases of [4].

One of the simplest unilateral source cooperative models is the *non-causal cognitive interference channel* [13]. This channel is similar to the classical IC with the difference that CTx has a full *a priori* / *non-causal* knowledge of the primary message. For this channel model the capacity region is known exactly for some parameter regimes and to within 1 bit otherwise [14]. In this paper we remove this ideal assumption, considering a more realistic scenario, where CTx learns the message of PTx through a noisy channel. An interesting question we answer with this work is when causal cooperation achieves the same Degrees-of-Freedom (DoF) as non-causal message knowledge.

B. Contributions

This work characterizes both the *symmetric* and *asymmetric* generalized DoF, or simply DoF for short in the rest of the paper, of the Gaussian CCIC (G-CCIC). The symmetric sum-rate is the maximum sum-capacity that the PTx and CTx can

achieve in a network with equal interfering links, while the asymmetric one considers the case where one interfering link is zero. Our main contributions are:

- 1) We identify simple achievable schemes, from the very general but highly complex scheme of [4], for the different parameter regimes that achieve the DoF upper bounds to within a constant gap. These schemes can be used as guidelines to deploy practical cognitive radio systems.
Interestingly, most of the schemes only uses superposition coding. The only exception is for small set of parameters in the S-channel where we were not able to show optimality to within a constant gap with the simple superposition coding scheme that was proposed for the interference-symmetric channel.
Moreover, we provide an example where both superposition coding and binning/dirty-paper-coding are optimal to within a constant gap. We show that binning/dirty-paper-coding achieves a smaller gap than superposition coding. Considering that that binning/dirty-paper-coding is more complex than superposition coding, our example points to an interesting practical trade-off between complexity and constant gap.
- 2) We explicitly identify the parameter regimes where causal cognitive radio offers unbounded gain (i.e., strictly larger DoF) with respect to the non-cooperative case. We also identify the regimes where causal cognitive radio achieves the same DoF as the idealized non-causal cognitive model.
- 3) We also develop a baseline scheme by time-sharing between the achievable inner bound of the IC and the one of the Relay Channel (RC). This simple strategy can be exploited to show the achievability of the degrees-of-freedom upper bound in the regimes where causal unilateral cooperation achieves the same DoF as the IC or the RC.

C. Paper Organization

The rest of the paper is organized as follows. Section II describes the channel model, defines the concept of DoF and illustrates known outer-bounds. Section III reports two simple different achievable schemes for the Gaussian CCIC. Section IV shows that the symmetric and asymmetric DoF can be achieved to within a constant gap irrespectively of the channel parameters. Section V concludes the paper.

II. SYSTEM MODEL AND BACKGROUND

A CCIC consists of two input alphabets $(\mathcal{X}_p, \mathcal{X}_c)$, three outputs alphabets $(\mathcal{Y}_f, \mathcal{Y}_p, \mathcal{Y}_c)$ and a memoryless transition probability $\mathcal{P}_{Y_f, Y_p, Y_c | X_p, X_c}$. PTx has a message $W_p \in [1 : 2^{NR_p}]$ for PRx and CTx has a message $W_c \in [1 : 2^{NR_c}]$ for CRx, where N denotes the codeword length and R_p and R_c the transmission rates for PTx and CTx, respectively. The messages W_p and W_c are independent and uniformly distributed on their respective domains. At time $i \in [1 : N]$ the PTx maps its message W_p into a channel input symbol $X_{p,i}(W_p)$ and CTx maps its message W_c and its past channel observations into a channel input symbol $X_{c,i}(W_c, Y_f^{i-1})$. At time N , the PRx outputs an estimate of its intended message W_p based on all its channel observations as $\widehat{W}_p(Y_p^N)$, and similarly CRx outputs $\widehat{W}_c(Y_c^N)$. The capacity region is defined as the convex closure of all non-negative rate pairs (R_p, R_c) such that $\max_{u \in \{c, p\}} \mathbb{P}[\widehat{W}_u \neq W_u] \rightarrow 0$ as $N \rightarrow \infty$.

A. The Gaussian Noise Channel and Generalized Degrees of Freedom

A single-antenna full-duplex G-CCIC, shown in Fig. 1, is described by the input/output relationship

$$Y_j = h_{pj}X_p + h_{cj}X_c + Z_j \quad j \in \{p, c, f\}, \quad (1)$$

where the channel gains are complex-valued, constant, and therefore known to all terminals. The channel inputs are subject to the power constraints $\mathbb{E}[|X_i|^2] \leq P_i \in \mathbb{R}^+, i \in \{p, c\}$. We assume without loss of generality that $Z_k \sim \mathcal{CN}(0, 1)$, $k \in \{f, p, c\}$. We could further set without loss of generality either $P_p = P_c = 1$ or $h_{pp} = h_{cc} = 1$ [8]. In the following we assume that the noises are independent.

A G-CCIC is said to be a Z-channel if $h_{pc} = 0$, i.e., the CRx does not experience interference from PTx and an S-channel if $h_{cp} = 0$, i.e., the PRx does not experience interference from CTx.

An often adopted figure of merit for the Gaussian channel is DoF defined as follows. Let $S > 1$ and parameterize

$$P_p|h_{pp}|^2 = P_c|h_{cc}|^2 := S^1 = S, \quad (2a)$$

$$P_p|h_{pc}|^2 := S^{\alpha_c} = I_c, \quad \alpha_c \geq 0, \quad (2b)$$

$$P_c|h_{cp}|^2 := S^{\alpha_p} = I_p, \quad \alpha_p \geq 0, \quad (2c)$$

$$P_p|h_{pf}|^2 := S^{\beta_c} = C_c, \quad \beta_c \geq 0, \quad (2d)$$

$$P_c|h_{cf}|^2 := S^{\beta_p} = C_p, \quad \beta_p \geq 0, \quad (2e)$$

where α_c and α_p measure the strength of the interference/cross links compared to the direct link, while β_c and β_p the strength of the cooperation links compared to the direct link. Since our setting corresponds to $C_p = 0$, we let for brevity $C_c = C$. The *Generalized Degrees of Freedom* of a Gaussian noise channel is defined as [8], [7]

$$d(\alpha_i, \alpha_f) := \lim_{S \rightarrow +\infty} \frac{\max\{R_p + R_c\}}{2 \log_2(1 + S)} \quad (3)$$

where the maximization is intended over all possible achievable rate pairs (R_c, R_p) .

In this work we focus on the DoF of both the symmetric and asymmetric G-CCIC, i.e. Z-channel and S-channel:

$$\begin{aligned} \text{interference-symmetric channel : } & \alpha_c = \alpha_i, \alpha_p = \alpha_i, \beta_c = \alpha_f, \beta_p = 0, \\ \text{Z-channel : } & \alpha_c = 0, \alpha_p = \alpha_i, \beta_c = \alpha_f, \beta_p = 0, \\ \text{S-channel : } & \alpha_c = \alpha_i, \alpha_p = 0, \beta_c = \alpha_f, \beta_p = 0. \end{aligned}$$

B. Known Outer-Bounds

In the literature several outer-bounds are known for bilateral source cooperation [5], [7], [11], [9]. Here we specialize them for the case of unilateral cooperation on the complex-valued Gaussian channel with independent noises. We define $\mathbb{E}[X_p X_c^*] = \rho \sqrt{P_p P_c}$ for some $\rho \in \mathbb{C}$ such that $|\rho| \leq 1$. The outer-bounds we will use are obtained by upper bounding each individual mutual information term over ρ in the bounds derived in [5], [7], [11]. In particular (see also Appendix B) the following bounds suffice to characterize the DoF of the G-CCIC to within a constant gap: CS Outer-Bound [5] in (4), DT Outer-Bound [11] in (5) and PV Outer-Bound [7] in (6).

$$R_p + R_c \leq \log_2(1 + S) + \min \left\{ \log_2 \left(1 + (\sqrt{S} + \sqrt{I_p})^2 \right), \log_2(1 + C + S) \right\}, \quad (4)$$

$$\begin{aligned} R_p + R_c \leq \min \left\{ \log_2 \left(\frac{1 + \max\{I_p, S\}}{1 + I_p} \right) + \log_2 \left(1 + (\sqrt{S} + \sqrt{I_p})^2 \right), \right. \\ \left. \log_2 \left(\frac{1 + C + \max\{I_c, S\}}{1 + I_c} \right) + \log_2 \left(1 + (\sqrt{S} + \sqrt{I_c})^2 \right) \right\}, \end{aligned} \quad (5)$$

$$R_p + R_c \leq \log_2 \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I_c}} + \sqrt{I_p} \right)^2 \right) + \log_2(1 + C) + \log_2 \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I_p}} + \frac{\sqrt{I_c}}{\sqrt{C}} \right)^2 \right). \quad (6)$$

III. BASELINE ACHIEVABLE STRATEGIES

If the CTx fully cooperates in sending the message of PTx without sending any information for itself, the channel reduces to a classical RC. The largest achievable rate for the RC is the combination of Partial-Decode-Forward (PDF) and Compress-Forward (CF) proposed in the seminal work of Cover and ElGamal [15]. Let $\mathcal{R}^{(\text{RC})}$ denote the achievable region in this case. If the CTx does not cooperate at all in sending the message of PTx, the channel reduces to a classical IC. The largest achievable region in this case is given by the Han and Kobayashi region [2] and denoted by $\mathcal{R}^{(\text{IC})}$. Here we propose two baseline schemes which are obtained by time-sharing between $\mathcal{R}^{(\text{RC})}$ and $\mathcal{R}^{(\text{IC})}$.

A. Baseline PDF-based Strategy

Instead of the general region $\mathcal{R}^{(\text{RC})}$ we consider here the achievable rate with PDF only, where the CTx acts as a relay for the primary pair, given by [16, Th.16.3]. We evaluate the PDF achievable rate with jointly Gaussian inputs, which are optimal to within 1bit for the full-duplex Gaussian RC [8], and indicate the corresponding achievable region as $\mathcal{R}^{(\text{PDF})}(P_p, P_c)$. Similarly, instead of the general region $\mathcal{R}^{(\text{IC})}$ we consider the achievable rate region in [2] with jointly Gaussian inputs, without time sharing and with the special rate-split of [8], which is optimal to within 1bit for the Gaussian IC [8], and indicate the corresponding achievable region as $\mathcal{R}^{(\text{ETW})}(P_p, P_c)$. By time-sharing between these two strategies we achieve the region in (7) where union is over $(P_{p1}, P_{p2}, P_{c1}, P_{c2}, \tau) \in \mathbb{R}_+^5$ such that for fixed average power constraint (P_p, P_c) the constraint in (8) is satisfied

$$\begin{aligned} \mathcal{R}^{(\text{Baseline})} := \bigcup \left\{ (R_p, R_c) \in \mathbb{R}_+^2 : R_c \leq \tau a_1 + (1 - \tau) a_2, R_p \leq \tau b_1 + (1 - \tau) b_2, \right. \\ \left. \text{with } (a_1, b_1) \in \mathcal{R}^{(\text{PDF})}(P_{p1}, P_{c1}), (a_2, b_2) \in \mathcal{R}^{(\text{ETW})}(P_{p2}, P_{c2}) \right\}, \end{aligned} \quad (7)$$

$$\tau P_{j1} + (1 - \tau) P_{j2} \leq P_j, \quad j \in \{p, c\}, \quad \text{and with } \tau \in [0, 1]. \quad (8)$$

Note that τ represents the fraction of time the CTx operates as a pure relay for the PTx.

In particular, since the PDF lower bound for the Gaussian RC is equal to the maximum of the decode-forward lower bound and the direct transmission lower bound [16, Remark 16.2] we write $\mathcal{R}^{(\text{PDF})}(P_p, P_c)$ as in (9).

$$\mathcal{R}^{(\text{PDF})}(P_p, P_c) := \left\{ R_c = 0, 0 \leq R_p \leq \max \left\{ \log_2(1 + S), r^{(\text{PDF})}(P_p, P_c) \right\} \right\} \quad (9)$$

$$r^{(\text{PDF})}(P_p, P_c) := \begin{cases} \log_2 \left(1 + \left(\sqrt{S \left(1 - \frac{I_p}{C} \right)} + \sqrt{I_p \left(1 - \frac{S}{C} \right)} \right)^2 \right) & \text{if } C \geq S + I_p \\ \log_2(1 + C) & \text{otherwise.} \end{cases} \quad (10)$$

The ETW region $\mathcal{R}^{(\text{ETW})}(P_p, P_c)$ is given by [8], [17] and is reported in Appendix A for completeness.

B. Baseline NNC-based Strategy

The baseline achievable rate region discussed above is based on PDF relaying at the CTx. However, PDF is not optimal in general. In particular, for the Gaussian RC, CF is known to outperform PDF when the relay is closer to the destination than to the source. Next we replace the PDF region in (7) with the recent network generalization of CF known as *Noisy Network Coding* (NNC) [18]. We apply [16, Th. 18.3] to the relay phase in our setting, in which the NNC lower bound for a memoryless network reduces to an equivalent formulation of the CF lower bound for the RC given by [16, Remark 16.4]. Without time sharing, with Gaussian inputs and with the optimal ‘noise quantization’ variance [16, eq.(16.12)], the achievable region is formally as in (7), optimized over the set of parameters defined in (8), but where instead of $\mathcal{R}^{(\text{PDF})}(P_p, P_c)$ we use $\mathcal{R}^{(\text{NNC})}(P_p, P_c)$ defined as

$$\mathcal{R}^{(\text{NNC})}(P_p, P_c) := \left\{ R_c = 0, 0 \leq R_p \leq \log_2 \left(1 + S + \frac{I_p C}{S + I_p + C + 1} \right) \right\}. \quad (11)$$

C. Numerical Evaluation

Fig. 2 compares the two baseline schemes described above for the symmetric G-CCIC where the scenario features strong cooperation ($\alpha_f = 1.2$) under different interfering levels at the receivers. The reported outer-bound is found by taking the intersection of [5], [7], [11], [9] for each value of the input correlation coefficient ρ .

From Fig. 2 we notice that by increasing the level of interference the rate of the primary pair R_p increases. This is due to the fact that, in weak interference, PTx prefers to communicate with PRx through the direct link since this leads to higher rates. On the other hand, the PTx reaches higher rates when the regarded scenario features both strong cooperation and high interference. In this situation indeed the performance in terms of primary user’s rates will depend on $\min \{|h_{pf}|, |h_{cp}|\}$ and not anymore on the direct link. From Fig. 2 we also observe that in the strong cooperation regime the NNC strategy outperforms the PDF one when the channel between CTx and PRx (h_{cp}) is sufficiently strong, while in the other cases the opposite holds. As we shall see later, these baseline strategies can be used whenever the DoF of the CCIC are as for the IC or for the RC.

IV. DoF AND CAPACITY TO WITHIN A CONSTANT GAP

Our main result is to show that the sum-rate upper bounds summarized in Section II-B are achievable to within a constant gap for the interference-symmetric and the interference-asymmetric cases.

The upper bounds in (4), (5) and (6) imply

$$2 \, d(\alpha_c, \alpha_p, \alpha_f) \leq \min \left\{ \begin{aligned} & d^{(\text{CS})}(\alpha_p, \beta_c) + d^{(\text{CS})}(\alpha_c, \beta_p), \\ & \min \{ d^{(\text{DT})}(\alpha_p, \beta_c), d^{(\text{DT})}(\alpha_c, \beta_p) \}, \\ & d^{(\text{PV})}(\alpha_c, \alpha_p, \beta_c, \beta_p) \end{aligned} \right\}, \quad (12a)$$

$$\min \{ d^{(\text{DT})}(\alpha_p, \beta_c), d^{(\text{DT})}(\alpha_c, \beta_p) \}, \quad (12b)$$

$$d^{(\text{PV})}(\alpha_c, \alpha_p, \beta_c, \beta_p) \}, \quad (12c)$$

where

$$d^{(\text{CS})}(\alpha, \beta) := \max \{ 1, \min \{ \alpha, \beta \} \}, \quad (12d)$$

$$d^{(\text{DT})}(\alpha, \beta) := \max \{ \beta, \alpha, 1 \} - \alpha + \max \{ \alpha, 1 \}, \quad (12e)$$

$$d^{(\text{PV})}(\alpha_c, \alpha_p, \beta_c, \beta_p) := \max \{ 1 - \alpha_c + \beta_p, \alpha_p \} + \max \{ 1 - \alpha_p + \beta_c, \alpha_c \}. \quad (12f)$$

The proof of (12) can be found in Appendix B.

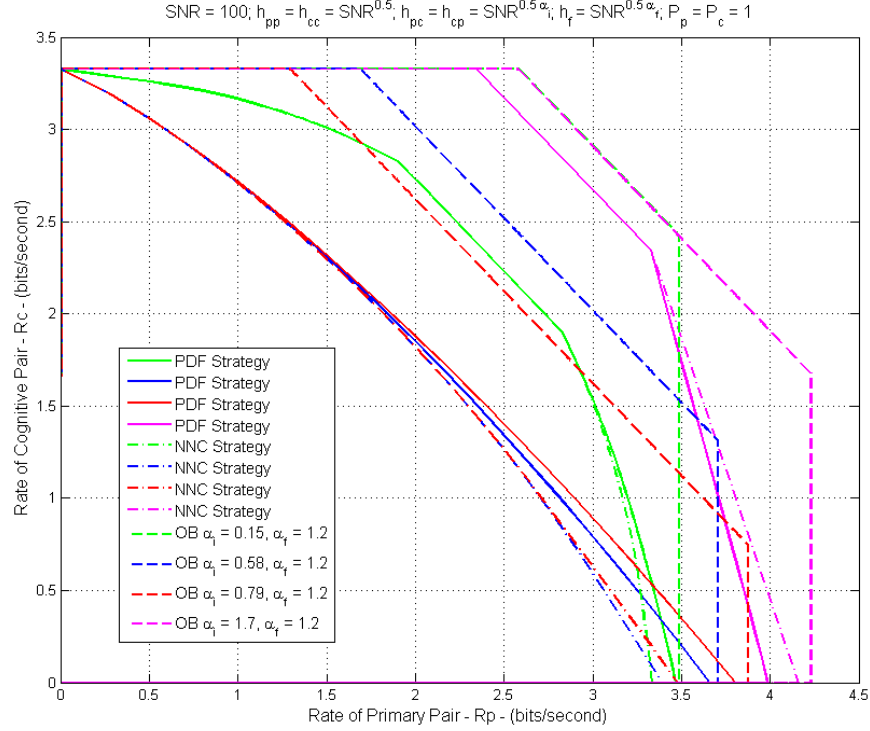


Fig. 2. Baseline PDF- and NNC-based strategies for a fixed α_f and for several values of α_i covering the noisy interference (green), very weak (blue), moderately weak (red) and strong (magenta) interference regimes.

A. Symmetric Channel

The DoF upper bound for the symmetric channel is obtained by setting $\alpha_c = \alpha_p = \alpha_i$, $\beta_c = \alpha_f$ and $\beta_p = 0$ in (12). Fig. 3 shows DoF and the gap for the symmetric G-CCIC. The whole set of parameters has been partitioned into multiple sub-regions depending upon different levels of cooperation (α_f) and interference (α_i) strengths. These regimes are numbered from 1 to 6 and the details for i -th region appear in i -th item in Appendix C. Our main result, proved in Appendix C, is:

Theorem 1 *The DoF upper bound in (12) is achievable to within 7.3 bits per channel use regardless of the actual value of the channel parameters for the symmetric G-CCIC.*

Few remarks before concluding this section:

- The gap result in item 6 / “Weak Interference 2” in Appendix C also holds for a big part of the regime “Moderately Weak Interference”. In particular, it does not apply in the regime $\alpha_i \in [2/3, 1]$, $\alpha_f < 1 - \alpha_i$ where the proposed scheme for “Weak Interference 2” does not achieve the optimal DoF.
- The largest gap in “Weak Interference 2” regime is 5 bits. This gap may be decreased in several ways. For example, one can optimize the power split between common and private message instead of using the one inspired by [8]. Alternatively, one can develop more complex coding scheme. An example of latter method can be found in Appendix H-A, where we consider a DPC-based achievable scheme. Then in Appendix H-B we use this scheme for the regime $\alpha_i < 1$ and $\alpha_f > 1$ and show it achieves the optimal DoF within 2 bits, rather than 3 bits as the previous proposed scheme based on superposition coding only. The DPC-based scheme, although achieving a smaller gap, is more complex to implement in practice than superposition coding.
- The largest gap occurs when the PV upper bound is the tightest. A possible way to reduce the gap would be to develop a tighter upper bound than the one used here.
- Cooperation does not improve on DoF of the classical IC when $\alpha_i \geq \frac{1}{2}$ and $\alpha_f \leq \min\{1, [2\alpha_i - 1]^+\}$. In this case our baseline strategies with $\tau = 0$ achieve the DoF upper bound.
- Similarly, cooperation does not improve on the classical RC when $\alpha_i \geq 2$ and $\alpha_f \geq \alpha_i$. In this case our baseline strategies with $\tau = 1$ achieve the DoF upper bound.
- The DoF of the G-CCIC are equal to those of the non-causal cognitive IC when

$$\max\{\alpha_i, 1 - \alpha_i\} \leq \alpha_f \leq \min\{1, 2\alpha_i - 1, 1 - \alpha_i\}.$$

B. Z-channel

In this section we consider the asymmetric-interference scenario where the PRx does not experience interference. The DoF upper bound for the Z-channel is obtained by setting $\alpha_c = 0$, $\alpha_p = \alpha_i$, $\beta_c = \alpha_f$ and $\beta_p = 0$ in (12). Fig. 4 shows DoF and the gap for the Z-channel. The whole set of parameters has been partitioned into multiple sub-regions depending upon different levels of cooperation (α_f) and interference (α_i) strengths. Our main result, whose proof is in Appendix D is:

Theorem 2 *The DoF upper bound in (12) is achievable to within 1 bit per channel use regardless of the actual value of the channel parameters for the Z-channel.*

Note that for the non-cooperative IC, it is well known that removing an interference link cannot degrade performance and the sum-capacity is known exactly for all channel parameters [19]. The same can not be said in full generality for the cooperative channel because “useful cooperative information” can flow through the interference link. For the Z-channel:

- By ignoring cooperation we immediately have $d(\alpha_i, \alpha_f) \geq \min\{1, \max\{\frac{\alpha_i}{2}, 1 - \frac{\alpha_i}{2}\}\}$ from [19, Th.2]. Thus for the Z-channel, cooperation only improves the DoF with respect to the non-cooperative case in the regime $\alpha_i \geq 2$ and $\alpha_f \geq 1$.
- In some regions the Z-channel outperforms the symmetric channel. This happens in weak interference when $0 \leq \alpha_i \leq \frac{2}{3}$, $\alpha_f \leq \alpha_i$ and $\alpha_f \leq 1 - \alpha_i$ as a result of not having interference from the PTx since $h_{pc} = 0$.
- By comparing Fig. 3 and Fig. 4 we observe that the DoF of the Z-channel are always greater than those of the symmetric G-CCIC. This is due to the fact that the PTx does not cooperate in sending the cognitive signal. Therefore by removing the link between PTx and CRx we rid CRx of only interfering signal and this leads to an improvement in DoF.
- The Z-channel achieves the same DoF of the non-causal cognitive channel, which are give by $d = \max\{1, \alpha_i\} - \alpha_i/2$, everywhere except in $\alpha_i > 2$.

C. S-channel

In this section we consider the asymmetric-interference scenario where the CRx does not experience interference. The DoF upper bound for the S-channel is obtained by setting $\alpha_c = \alpha_i$, $\alpha_p = 0$, $\beta_c = \alpha_f$ and $\beta_p = 0$ in (12). Fig. 5 shows DoF and the gap for the S-channel. The whole set of parameters has been partitioned into multiple sub-regions depending upon different levels of cooperation (α_f) and interference (α_i) strengths. Our main result, whose proof is in Appendix E is:

Theorem 3 *The DoF upper bound in (12) is achievable to within 4 bits per channel use regardless of the actual value of the channel parameters for the S-channel.*

Few remarks:

- Cooperation improves the DoF with respect to the IC only when $\alpha_i \leq 2$ and $\alpha_f \geq \max\{2, 1 + \alpha_i\}$.
- The S-channel outperforms the interference-symmetric channel when either $0 \leq \alpha_i \leq \frac{2}{3}$, $\alpha_f \leq \alpha_i$ and $\alpha_f \leq 1 - \alpha_i$ or when $\alpha_i \leq 2$ and $\alpha_f \geq \max\{1, \alpha_i\}$.
- The interference-symmetric channel outperforms the S-channel in very strong interference and strong cooperation, i.e., $\alpha_i \geq 2$ and $\alpha_f \geq 1$. This is due to the fact that the information for the PRx can no longer be routed through the CTx since $h_{cp} = 0$.
- The S-channel achieves the same DoF of the non-causal cognitive channel ($d = 1$) everywhere except in $\alpha_i < 2$, $\alpha_f < \min\{1, \alpha_i\}$.
- When $\alpha_i \geq 2$, i.e. very strong interference regime, we have an exact sum-capacity result, i.e., the gap between the outer bound and inner bound is equal to zero.

V. CONCLUSIONS

In this work we considered the CCIC, a network with two source-destination pairs sharing the same channel. In contrast to the classical IC, in the CCIC the CTx exploits information about the PTx from its own channel observations. This scenario represents a more practically relevant model for Cognitive Radio than the non-causal cognitive IC, where the CTx is assumed to have a priori non-causal knowledge of the PTx’s message. In particular, it is applicable in practical relaying architectures for 4G cellular networks.

We developed and evaluated a baseline simple achievable scheme based on time-sharing between the achievable regions of the classical interference and relay channel; we showed by numerical evaluations that this simple strategy may be not too far from an outer bound in certain parameter regimes.

We exploited known outer bounds for bilateral source cooperation adapting them to the case of unilateral cooperation. Our main contribution consisted in showing that both for the interference-symmetric and interference-asymmetric cases the sum-capacity of the two-user Gaussian IC with independent noises can be achieved to within a constant gap. Interestingly, the achievable schemes only use superposition coding except for a small set of parameters in the S-channel. It is shown that

more complex schemes employing binning/dirty-paper-coding could be used to achieve a smaller gap at the cost of increased coding complexity.

We also identified the set of parameters where causal cooperation achieves the same DoF as the classical interference channel (meaning that cooperation in this case only offers “beam-forming gains”) and the set of parameters where it achieves the same DoF as the non-causal cognitive interference channel.

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APPENDIX A
ETW REGION FOR THE INTERFERENCE CHANNEL

The ETW region $\mathcal{R}^{(\text{ETW})}(P_p, P_c)$ is given by [17]

$$\begin{aligned} \mathcal{R}^{(\text{ETW})}(P_p, P_c) := & \left\{ (R_p, R_c) \in \mathbb{R}_+^2 : \right. \\ & R_p \leq \psi_1 := \log_2 \left(1 + \frac{S}{1 + \rho_c I_p} \right), \\ & R_c \leq \psi_2 := \log_2 \left(1 + \frac{S}{1 + \rho_p I_c} \right), \\ & R_p + R_c \leq \min\{\psi_3, \psi_4, \psi_5, \psi_1 + \psi_2, (\psi_6 + \psi_2)/2, (\psi_7 + \psi_1)/2, (\psi_6 + \psi_7)/3\}, \\ & 2R_p + R_c \leq \psi_6 := \log_2 \left(1 + \frac{\rho_c S + (1 - \rho_p) I_c}{1 + \rho_p I_c} \right) \\ & \quad + \log_2 \left(1 + \frac{S + (1 - \rho_c) I_p}{1 + \rho_c I_p} \right) + \log_2 \left(1 + \frac{\rho_p S}{1 + \rho_c I_p} \right), \\ & R_p + 2R_c \leq \psi_7 := \log_2 \left(1 + \frac{\rho_p S + (1 - \rho_c) I_p}{1 + \rho_c I_p} \right) \\ & \quad + \log_2 \left(1 + \frac{S + (1 - \rho_p) I_c}{1 + \rho_p I_c} \right) + \log_2 \left(1 + \frac{\rho_c S}{1 + \rho_p I_c} \right), \\ & \psi_3 := \log_2 \left(1 + \frac{\rho_c S}{1 + \rho_p I_c} \right) + \log_2 \left(1 + \frac{S + (1 - \rho_c) I_p}{1 + \rho_c I_p} \right), \\ & \psi_4 := \log_2 \left(1 + \frac{S + (1 - \rho_p) I_c}{1 + \rho_p I_c} \right) + \log_2 \left(1 + \frac{\rho_p S}{1 + \rho_c I_p} \right), \\ & \psi_5 := \log_2 \left(1 + \frac{\rho_c S + (1 - \rho_p) I_c}{1 + \rho_p I_c} \right) \\ & \quad + \log_2 \left(1 + \frac{\rho_p S + (1 - \rho_c) I_p}{1 + \rho_c I_p} \right) \\ & \rho_p \in [0, 1], \\ & \left. \rho_c \in [0, 1], \right\} \end{aligned}$$

where ρ_p (respectively ρ_c) is the fraction of power the PTx (reps. CTx) allocates for its private message.

APPENDIX B
DOF UPPER BOUND

In order to prove (12) we use the known outer bounds of [5], [7], [11].

- The cut-set upper bound for a relay channel with gain S on the link from the source to the destination, gain C on the link from the source to the relay, and gain I on the link from the relay to the destination is

$$\begin{aligned} & \max_{|\rho| \leq 1} \min \left\{ \log_2 \left(1 + S + I + 2|\rho|\sqrt{SI} \right), \log_2 \left(1 + (1 - |\rho|^2) (C + S) \right) \right\} \\ & \leq \min \left\{ \log_2 \left(1 + (\sqrt{S} + \sqrt{I})^2 \right), \log_2 (1 + C + S) \right\} =: \mathcal{C}_{\text{RC}}(S, I, C), \end{aligned}$$

whose behavior at high SNR gives (12d).

The cut-set upper for the general cooperative interference channel bound is equivalent to

$$R_p \leq \mathcal{C}_{\text{RC}}(S, I_p, C_c), \quad R_c \leq \mathcal{C}_{\text{RC}}(S, I_c, C_p),$$

which implies (12a).

- From [11] (from which the DT upper bound in (5) was derived) we have

$$\begin{aligned} R_p + R_c & \leq \max_{|\rho| \leq 1} \log_2 \left(\frac{1 + (1 - |\rho|^2) (C_c + \max\{I_c, S\})}{1 + (1 - |\rho|^2) I_c} \right) + \\ & + \log_2 \left(1 + I_c + S + 2|\rho|\sqrt{SI_c} \right) \leq \log_2 \left(\frac{1 + C_c + \max\{I_c, S\}}{1 + I_c} \right), \\ & + \log_2 \left(1 + (\sqrt{I_c} + \sqrt{S})^2 \right) =: \mathcal{C}_{\text{DT}}(S, I_c, C_c) \end{aligned}$$

whose behavior at high SNR gives (12e). This also proves the bound in (4).
By swapping the role of the cognitive and primary pairs we have

$$R_p + R_c \leq \mathcal{C}_{\text{DT}}(S, I_p, C_p).$$

These two sum-rate bounds imply the upper bound in (12b).

- From [7] (from which the PV upper bound in (6) was derived) we have

$$\begin{aligned} R_p + R_c &\leq \log_2(1 + C_c) + \log_2(1 + C_p) \\ &+ \log_2 \left(1 + \left(\frac{\sqrt{S}}{\max\{1, \sqrt{I_c}\}} + \frac{\sqrt{I_p}}{\max\{1, \sqrt{C_p}\}} \right)^2 \right) \\ &+ \log_2 \left(1 + \left(\frac{\sqrt{S}}{\max\{1, \sqrt{I_p}\}} + \frac{\sqrt{I_c}}{\max\{1, \sqrt{C_c}\}} \right)^2 \right) \end{aligned}$$

whose behavior at high SNR gives (12f).

APPENDIX C

CONSTANT GAP RESULT FOR THE SYMMETRIC G-CCIC

We let $I_p = I_c = I$ for brevity and we analyze different regimes by developing simple achievable strategies for each.

- 1) **Very Strong Interference and Weak Cooperation:** $\alpha_i > 2$ and $\alpha_f \leq 1$.

Parameter Range: $I > S(1 + S)$ and $C \leq S$, in which case the tightest upper bound gives $d(\alpha_i, \alpha_f) \leq 1$.

Inner Bound: classical IC in very strong interference with only common messages,

$$(R_p + R_c)^{(\text{IB})} = 2 \log_2(1 + S),$$

which implies

$$d(\alpha_i, \alpha_f) \geq \lim_{S \rightarrow +\infty} \frac{(R_p + R_c)^{(\text{IB})}}{2 \log_2(1 + S)} = \frac{2 \log_2(1 + S)}{2 \log_2(1 + S)} = 1.$$

This shows the achievability of the DoF upper bound.

Outer Bound: from (4) by using $C \leq S$

$$\begin{aligned} (R_p + R_c)^{(\text{OB})} &\leq \log_2(1 + C + S) + \log_2(1 + S) \\ &\leq \log_2(1 + 2S) + \log_2(1 + S) \\ &\leq 2 \log_2(1 + S) + \log_2(2). \end{aligned}$$

Gap between OB and IB:

$$\text{GAP} = (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \leq 1 \text{ bit}.$$

- 2) **Very Strong Interference and Strong Cooperation:** $\alpha_i > 2$ and $\alpha_f > 1$.

Parameter Range: $I > S(1 + S)$ and $C > S$, in which case the tightest upper bound gives

$$d(\alpha_i, \alpha_f) \leq \frac{1}{2} \min\{\alpha_i, 1 + \alpha_f\}.$$

Inner Bound: In Appendix F we show that the following sum-rate is achievable with superposition coding and common messages only

$$(R_p + R_c)^{(\text{IB})} \geq \begin{cases} \log_2(1 + C) + \log_2(1 + S) & \text{if } C(1 + S) \leq I, \\ \log_2(1 + S + I) & \text{if } C(1 + S) > I, \end{cases}$$

which implies

$$\begin{aligned} d(\alpha_i, \alpha_f) &\geq \lim_{S \rightarrow +\infty} \frac{(R_p + R_c)^{(\text{IB})}}{2 \log_2(1 + S)} \\ &\geq \lim_{S \rightarrow +\infty} \frac{\min\{\log_2(1 + C) + \log_2(1 + S), \log_2(1 + S + I)\}}{2 \log_2(1 + S)} \\ &= \frac{1}{2} \min\{1 + \alpha_f, \max\{1, \alpha_i\}\} \\ &\stackrel{\text{since } \alpha_i \geq 1}{=} \frac{1}{2} \min\{1 + \alpha_f, \alpha_i\}. \end{aligned}$$

This shows the achievability of the DoF upper bound.

Outer Bound: for the case $S < C \leq I/(1+S)$, from (4) we have

$$(R_p + R_c)^{(\text{OB})} \leq \log_2(1 + C + S) + \log_2(1 + S),$$

while for the case $C > I/(1+S)$ from (5) (by using $S \leq I$) we have

$$(R_p + R_c)^{(\text{OB})} \leq 0 + \log_2(1 + S + I + 2\sqrt{SI}).$$

Gap between OB and IB: for the case $S < C \leq I/(1+S)$ we obtain

$$\begin{aligned} \text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\ &\leq \log_2(1 + C + S) + \log_2(1 + S) - \log_2(1 + C) - \log_2(1 + S) \\ &= \log_2\left(1 + \frac{S}{1 + C}\right) \\ &\leq \log_2\left(1 + \frac{S}{1 + S}\right) \leq \log_2(1 + 1) = 1 \text{ bit}, \end{aligned}$$

while for the case $C > I/(1+S)$ we obtain

$$\begin{aligned} \text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\ &\leq \log_2(1 + S + I + 2\sqrt{SI}) - \log_2(1 + S + I) \\ &= \log_2\left(1 + \frac{2\sqrt{SI}}{1 + S + I}\right) \leq \log_2(1 + 1) = 1 \text{ bit}, \end{aligned}$$

since $2\sqrt{SI} \leq 1 + S + I \iff 0 \leq 1 + (\sqrt{S} - \sqrt{I})^2$.

3) **Strong Interference:** $\alpha_i \in [1, 2]$.

Parameter Range: $S \leq I \leq S(1+S)$, in which case the tightest upper bound gives

$$d(\alpha_i, \alpha_f) \leq \frac{1}{2}\alpha_i.$$

Inner Bound: classical IFC in strong interference (HK with common messages only and joint decoding), that is, baseline TDM strategy with $\tau = 0$

$$(R_p + R_c)^{(\text{IB})} = \log_2(1 + S + I),$$

which implies

$$\begin{aligned} d(\alpha_i, \alpha_f) &\geq \lim_{S \rightarrow +\infty} \frac{(R_p + R_c)^{(\text{IB})}}{2 \log_2(1 + S)} \\ &= \lim_{S \rightarrow +\infty} \frac{\log_2(1 + I + S)}{2 \log_2(1 + S)} \\ &= \frac{1}{2} \max\{1, \alpha_i\} \\ &\stackrel{\text{since } \alpha_i \geq 1}{=} \frac{1}{2} \alpha_i. \end{aligned}$$

This shows that DoF upper bound is achievable.

Outer Bound: from (5) by using $S \leq I$

$$(R_p + R_c)^{(\text{OB})} \leq 0 + \log_2(1 + S + I + 2\sqrt{SI}).$$

Gap between OB and IB:

$$\begin{aligned} \text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\ &\leq \log_2(1 + S + I + 2\sqrt{SI}) - \log_2(1 + S + I) \leq 1 \text{ bit}, \end{aligned}$$

since $2\sqrt{SI} \leq 1 + S + I \iff 0 \leq 1 + (\sqrt{S} - \sqrt{I})^2$.

4) **Moderately Weak Interference:** $\alpha_i \in [2/3, 1]$

Parameter Range: $I < S$ and $S(S+I) \leq I^2(I+1)$, or equivalently $I < S \leq I\left(\sqrt{I + \frac{5}{4}} - \frac{1}{2}\right)$, in which case the tightest upper bound gives

$$d(\alpha_i, \alpha_f) \leq 1 - \frac{\alpha_i}{2}.$$

Inner Bound: classical IFC with common and private messages and with the power split of [8], that is, baseline TDM strategy with $\tau = 0$

$$(R_p + R_c)^{(\text{IB})} = \log_2(1 + S + I) + \log_2\left(2 + \frac{S}{I}\right) - 2,$$

which implies

$$\begin{aligned} d(\alpha_i, \alpha_f) &\geq \lim_{S \rightarrow +\infty} \frac{(R_p + R_c)^{(\text{IB})}}{2 \log_2(1 + S)} \\ &= \lim_{S \rightarrow +\infty} \frac{\log_2(1 + S + I) + \log_2\left(2 + \frac{S}{I}\right) - 2}{2 \log_2(1 + S)} \\ &= \frac{1}{2} \max \{ \max\{1, \alpha_i\}, [1 - \alpha_i]^+ \} \\ &\stackrel{\text{since } \alpha_i < 1}{=} \frac{2 - \alpha_i}{2}. \end{aligned}$$

This shows the achievability of the DoF upper bound.

Outer Bound: from (5) by using $I \leq S$

$$(R_p + R_c)^{(\text{OB})} \leq \log\left(\frac{1 + S}{1 + I}\right) + \log\left(1 + (\sqrt{S} + \sqrt{I})^2\right).$$

Gap between OB and IB:

$$\begin{aligned} \text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\ &\leq \log_2\left(\frac{1 + S}{1 + I}\right) + \log_2\left(1 + (\sqrt{S} + \sqrt{I})^2\right) + \\ &\quad - \log_2(1 + S + I) - \log_2\left(2 + \frac{S}{I}\right) + 2 \\ &\leq \log_2\left(1 + \frac{2\sqrt{SI}}{1 + S + I}\right) + \log_2\left(\frac{I(1 + S)}{S(1 + I)}\right) + 2 \\ &\leq \log_2(2) + \log_2(1) + 2 = 3 \text{ bits}, \end{aligned}$$

since $2\sqrt{SI} \leq 1 + S + I \iff 0 \leq 1 + (\sqrt{S} - \sqrt{I})^2$ and $I \leq S$.

5) **Weak Interference 1:** $\alpha_i \in [1/2, 2/3)$ and $\alpha_f \leq 2\alpha_i - 1$.

Parameter Range: $S(S + I) > I^2(I + 1)$ and $C \leq \frac{I^2}{S}$ and $I^2 \geq S$, or equivalently $I\left(\sqrt{I + \frac{5}{4}} - \frac{1}{2}\right) < S \leq I^2$ and $C \leq \frac{I^2}{S}$, in which case the tightest upper bound gives

$$d(\alpha_i, \alpha_f) \leq \alpha_i.$$

Inner Bound: classical IFC with common and private messages and with the power split of [8], that is, baseline TDM strategy with $\tau = 0$

$$(R_p + R_c)^{(\text{IB})} = 2 \log_2\left(1 + I + \frac{S}{I}\right) - 2,$$

which implies

$$\begin{aligned} d(\alpha_i, \alpha_f) &\geq \lim_{S \rightarrow +\infty} \frac{(R_p + R_c)^{(\text{IB})}}{2 \log_2(1 + S)} \\ &= \lim_{S \rightarrow +\infty} \frac{2 \log_2\left(1 + I + \frac{S}{I}\right) - 2}{2 \log_2(1 + S)} \\ &= \max\{\alpha_i, [1 - \alpha_i]^+\} \\ &\stackrel{\text{since } \alpha_i \in [1/2, 1]}{=} \alpha_i. \end{aligned}$$

This shows the achievability of the DoF upper bound.

Outer Bound: from (6)

$$\begin{aligned}
(R_p + R_c)^{(\text{OB})} &\leq \log_2 \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I}} + \sqrt{I} \right)^2 \right) + \\
&\quad + \log_2(1 + C) + \log_2 \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I}} + \frac{\sqrt{I}}{\sqrt{C}} \right)^2 \right) \\
&\leq 2 \log_2 \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I}} + \sqrt{I} \right)^2 \right) + \Delta, \\
\Delta &:= \max_{C \in [1, I^2/S]} \log_2 \frac{(1 + C) \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I}} + \frac{\sqrt{I}}{\sqrt{C}} \right)^2 \right)}{1 + \left(\frac{\sqrt{S}}{\sqrt{I}} + \sqrt{I} \right)^2} \leq \log_2(10).
\end{aligned}$$

The upper bound on Δ is derived as follow: as a function of C , Δ is a parabola with a minimum in C , therefore the maximum of Δ in $C \in [1, I^2/S]$ is attained either for $C = 1$ or for $C = I^2/S$. For $C = 1$ we have

$$\Delta = \log_2(1 + 1) = 1 \text{ bit},$$

while $C = I^2/S$ we have

$$\begin{aligned}
\Delta &= \log_2 \left(\frac{\left(1 + \frac{I^2}{S} \right) \left(1 + 4\frac{S}{I} \right)}{1 + \frac{S}{I} + I + 2\sqrt{S}} \right) = \\
&= \log_2(S + I^2) + \log_2 \left(\frac{I}{S} + 4 \right) - \log_2(I + S + I^2 + 2I\sqrt{S}) \\
&\leq \log_2(2I^2) + \log_2 \left(\frac{1}{\sqrt{I + \frac{5}{4} - \frac{1}{2}}} + 4 \right) - \log_2(I^2) \\
&= \log_2(2) + \log_2 \left(\frac{1}{\sqrt{1 + \frac{5}{4} - \frac{1}{2}}} + 4 \right) = \log_2(10),
\end{aligned}$$

by using $I \left(\sqrt{I + \frac{5}{4} - \frac{1}{2}} \right) < S \leq I^2$ and $1 \leq I$.

Gap between OB and IB:

$$(R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \leq (2 + \Delta + 2)\text{bits} < 7.322 \text{ bits}.$$

6) Weak Interference 2: $\alpha_i \leq 2/3$ and $\alpha_f > [2\alpha_i - 1]^+$

Parameter Range: $S(S + I) > I^2(I + 1)$ and $C \geq \frac{I^2}{S}$. In order to find the tightest upper bound we need to split this region in different subregions, namely:

- 6a) $S < C(I + 1)$: here the tightest upper bound gives $d(\alpha_i, \alpha_f) \leq 1 - \frac{\alpha_i}{2}$;
- 6b) $S \geq C(I + 1)$ and $C \geq I$: here the tightest upper bound gives $d(\alpha_i, \alpha_f) \leq 1 - \frac{\alpha_i}{2}$;
- 6c) $S \geq C(I + 1)$, $I^2 \leq S$ and $C < I$: here the tightest upper bound gives $d(\alpha_i, \alpha_f) \leq 1 - \alpha_i + \frac{\alpha_f}{2}$;
- 6d) $S \geq C(I + 1)$, $I^2 > S$, $C < I$ and $S(S + I) > I^2(I + 1)$: here the tightest upper bound gives $d(\alpha_i, \alpha_f) \leq \frac{1 + \alpha_f}{2}$.

Inner Bound: We use the achievable scheme in Appendix G to show that the following sum-rate is achievable with superposition coding with common and private messages

$$\begin{aligned}
(R_p + R_c)^{(\text{IB})} &\geq \min \left\{ \log_2 \left(1 + \frac{S}{2I} \right) + \log_2 \left(\frac{S + I + 1}{2} \right), \right. \\
&\quad \left. \log_2 \left(1 + \frac{S}{2I} \right) + \log_2 \left(\frac{1 + C}{I + C} \right) + \log_2 \left(\frac{S + I^2 + I}{2} \right) \right\}
\end{aligned}$$

is achievable.

The derived achievable rate in the weak interference regime (i.e., $\alpha_i \leq 1$) implies that

$$\begin{aligned}
d(\alpha_i, \alpha_f) &\geq \lim_{S \rightarrow \infty} \frac{(R_p + R_c)^{(\text{IB})}}{2 \log_2(1 + S)} \\
&= \frac{\min\{[1 - \alpha_i]^+ + \max\{1, \alpha_i\}, [1 - \alpha_i]^+ + \alpha_f - \max\{\alpha_i, \alpha_f\} + \max\{1, 2\alpha_i\}\}}{2} \\
&= \frac{\min\{[1 - \alpha_i]^+ + \max\{1, \alpha_i\}, [1 - \alpha_i]^+ + \min\{\alpha_i, \alpha_f\} + \max\{1 - \alpha_i, \alpha_i\}\}}{2} \\
&= \begin{cases} 1 - \alpha_i/2 & \text{for } 1 < \alpha_i + \alpha_f, \\ 1 - \alpha_i/2 & \text{for } 1 \geq \alpha_i + \alpha_f, \alpha_f \geq \alpha_i, \\ 1 - \alpha_i + \alpha_f/2 & \text{for } 1 \geq \alpha_i + \alpha_f, \alpha_f < \alpha_i, \alpha_i < 1/2, \\ (1 + \alpha_f)/2 & \text{for } 1 \geq \alpha_i + \alpha_f, \alpha_f < \alpha_i, \alpha_i \in [1/2, 1]. \end{cases}
\end{aligned}$$

This shows the achievability of the DoF upper bound in $\alpha_i \leq 2/3$.

Outer Bound: for the regimes where $d(\alpha_i, \alpha_f) \leq 1 - \alpha_i/2$ we use the upper bound in (5) under the weak interference condition $S \geq I$, that is

$$(R_p + R_c)^{(\text{OB})} \leq \log_2 \left(\frac{1 + S}{1 + I} \right) + \log_2 (1 + S + I + 2\sqrt{SI})$$

otherwise we use the upper bound in (6)

$$\begin{aligned}
(R_p + R_c)^{(\text{OB})} &\leq \log_2 \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I}} + \sqrt{I} \right)^2 \right) + \\
&\quad + \log_2 (1 + C) + \log_2 \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I}} + \frac{\sqrt{I}}{\sqrt{C}} \right)^2 \right).
\end{aligned}$$

Gap between OB and IB: we will analyze separately the different sub regimes.

gap 6a) For the regime $S < C(I + 1)$

$$\begin{aligned}
\text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\
&\leq \log_2 \left(\frac{1 + S}{1 + I} \right) + \log_2 (1 + S + I + 2\sqrt{SI}) - \log_2 \left(\frac{2I + S}{2I} \right) - \log_2 \left(\frac{S + I + 1}{2} \right) \\
&= 2 + \log_2 \left(\frac{1 + S}{2I + S} \cdot \frac{I}{1 + I} \right) + \log_2 \left(\frac{1 + S + I + 2\sqrt{SI}}{S + I + 1} \right) \\
&\leq 2 + \log_2(1 \cdot 1) + \log_2(2) = 3 \text{ bits,}
\end{aligned}$$

since $\frac{I}{1+I} \leq 1$, $\frac{1+S}{2I+S} \leq \max\{1, \frac{1}{2I}\} = 1$ because $I > 1$, and $2\sqrt{SI} \leq 1 + S + I \iff 0 \leq 1 + (\sqrt{S} - \sqrt{I})^2$ and $I < S$.

gap 6b) For the regime $S \geq C(I+1)$ and $C \geq I$

$$\begin{aligned}
\text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\
&\leq \log_2 \left(\frac{1+S}{1+I} \right) + \log_2 (1+S+I+2\sqrt{SI}) + \\
&\quad - \log_2 \left(\frac{2I+S}{2I} \right) - \log_2 \left(\frac{1+C}{I+C} \right) - \log_2 \left(\frac{S+I^2+I}{2} \right) \\
&= 2 + \log_2 \left(\frac{1+S}{1+I} \right) + \log_2 (1+S+I+2\sqrt{SI}) + \\
&\quad - \log_2 (2I+S) + \log_2 (I) - \log_2 \left(\frac{1+C}{I+C} \right) - \log_2 (S+I^2+I) \\
&\leq 2 + \log_2 \left(\frac{1+S}{1+I} \right) + \log_2 (1+S+I+2\sqrt{SI}) - \log_2 (2I+S) + \log_2 (I) + \\
&\quad - \log_2 \left(\frac{1+I}{2I} \right) - \log_2 (S+I^2+I) \\
&= 3 + \log_2 \left(\frac{1+S}{2I+S} \right) + 2\log_2 \left(\frac{I}{1+I} \right) + \log_2 \left(\frac{1+S+I+2\sqrt{SI}}{S+I^2+I} \right) \\
&\leq 3 + \log_2(1) + 2\log_2(1) + \log_2(2) = 4 \text{ bits,}
\end{aligned}$$

since $1+S+I < S+I^2+I$ and $2\sqrt{SI} < S+I^2+I$. Here we upper bounded the gap by evaluating it for $C = I$, i.e., minimum possible value for C , since the function is decreasing in C .

gap 6c) For the regime $S \geq C(I+1)$, $C < I$ and $I^2 \leq S$

$$\begin{aligned}
\text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\
&\leq \log_2 \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I}} + \sqrt{I} \right)^2 \right) + \log_2 (1+C) + \log_2 \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I}} + \frac{\sqrt{I}}{\sqrt{C}} \right)^2 \right) + \\
&\quad - \log_2 \left(\frac{2I+S}{2I} \right) - \log_2 \left(\frac{1+C}{I+C} \right) - \log_2 \left(\frac{S+I^2+I}{2} \right) \\
&= 2 + \log_2 (I+S+I^2+2I\sqrt{S}) - \log_2 (I) + \log_2 (1+C) + \\
&\quad + \log_2 (IC+SC+I^2+2I\sqrt{SC}) - \log_2 (IC) - \log_2 (2I+S) + \\
&\quad + \log_2 (I) - \log_2 (1+C) + \log_2 (I+C) - \log_2 (S+I^2+I) \\
&= 2 + \log_2 (I+S+I^2+2I\sqrt{S}) + \log_2 (IC+SC+I^2+2I\sqrt{SC}) + \log_2 (I+C) + \\
&\quad - \log_2 (IC) - \log_2 (2I+S) - \log_2 (S+I^2+I) \\
&\leq 3 + \log_2 (I+S+I^2+2I\sqrt{S}) + \log_2 (2I^2+SI+2I\sqrt{SI}) + \\
&\quad - \log_2 (I) - \log_2 (2I+S) - \log_2 (S+I^2+I) \\
&= 3 + \log_2 \left(\frac{I+S+I^2+2I\sqrt{S}}{S+I^2+I} \right) + \log_2 \left(\frac{2I^2+SI+2I\sqrt{SI}}{2I^2+SI} \right) \\
&= 3 + \log_2 \left(1 + \frac{2I\sqrt{S}}{S+I^2+I} \right) + \log_2 \left(1 + \frac{2I\sqrt{SI}}{2I^2+SI} \right) \\
&\leq 3 + \log_2(2) + \log_2(2) = 5 \text{ bits,}
\end{aligned}$$

since $2I\sqrt{S} < S+I^2+I$ and $2I < 2I^2+SI$. Here we upper bounded the gap by evaluating it for $C = I$, i.e., the maximum possible value for C , since the function is increasing in C .

gap 6d) For the regime $S \geq C(I+1)$, $C < I$, $I^2 > S$ and $S(S+I) \geq I^2(I+1)$

$$\begin{aligned} \text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\ &\leq \log_2 \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I}} + \sqrt{I} \right)^2 \right) + \log_2(1+C) + \log_2 \left(1 + \left(\frac{\sqrt{S}}{\sqrt{I}} + \frac{\sqrt{I}}{\sqrt{C}} \right)^2 \right) + \\ &\quad - \log_2 \left(\frac{2I+S}{2I} \right) - \log_2 \left(\frac{1+C}{I+C} \right) - \log_2 \left(\frac{S+I^2+I}{2} \right) \\ &\leq 5 \text{ bits,} \end{aligned}$$

by following exactly the same steps as done for $S > C(I+1)$, $C < I$ and $I^2 \leq S$ (case above).

This demonstrates the achievability of the DoF in (12) to within a constant gap of 7.3 bits/s/Hz.

APPENDIX D

CONSTANT GAP RESULT FOR THE Z-CHANNEL G-CCIC

We let $I_p = I$ for brevity and we analyze separately the weak and the strong regimes.

1) **Strong and Very Strong Interference:** $\alpha_i \geq 1$.

The analysis is similar to that of the channel with symmetric interfering links in the same regime (we use the same achievable strategies) and the gap is at most 1 bit.

2) **Weak Interference:** $\alpha_i < 1$.

Parameter Range: $I < S$, in which case the tightest upper bound gives

$$d(\alpha_i, \alpha_f) \leq 1 - \frac{1}{2}\alpha_i.$$

Inner Bound: classical Z-IFC in weak interference [19, Theorem 2],

$$(R_p + R_c)^{(\text{IB})} = \log_2(1+S) + \log_2 \left(1 + \frac{S}{1+I} \right),$$

which implies

$$\begin{aligned} d(\alpha_i, \alpha_f) &\geq \lim_{S \rightarrow +\infty} \frac{(R_p + R_c)^{(\text{IB})}}{2 \log_2(1+S)} \\ &= \frac{1 + [1 - \alpha_i]^+}{2} \\ &\stackrel{\text{since } \alpha_i < 1}{=} 1 - \frac{1}{2}\alpha_i. \end{aligned}$$

This shows that DoF upper bound is achievable.

Outer Bound: from Tuninetti's upper bound by using $I \leq S$

$$(R_p + R_c)^{(\text{OB})} \leq \log_2 \left(\frac{1+S}{1+I} \right) + \log_2 \left(1 + S + I + 2\sqrt{SI} \right).$$

Gap between OB and IB:

$$\begin{aligned} \text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\ &\leq \log_2 \left(\frac{1+S}{1+I} \right) + \log_2 \left(1 + S + I + 2\sqrt{SI} \right) + \\ &\quad - \log_2(1+S) - \log_2 \left(1 + \frac{S}{1+I} \right) \\ &= \log_2 \left(\frac{1+S+I+2\sqrt{SI}}{1+S+I} \right) \leq 1 \text{ bit,} \end{aligned}$$

since $2\sqrt{SI} \leq 1+S+I \iff 0 \leq 1 + (\sqrt{S} - \sqrt{I})^2$.

This demonstrates the achievability of the DoF in (12) to within a constant gap of 1 bit/s/Hz.

APPENDIX E
CONSTANT GAP RESULT FOR THE S-CHANNEL G-CCIC

We let $I_c = I$ for brevity and we analyze separately different regimes.

1) **Regime 1:** $\{\alpha_i \in [0, 1], \alpha_f > 1\} \cup \{\alpha_i \in [1, 2], \alpha_f > 1 + \alpha_i\}$.

With the DPC-based achievable scheme from Appendix H-A the achievable sum-rate, after setting the appropriate interfering link to zero, is

$$\begin{aligned} (R_p + R_c)^{(\text{IB})} &= \max_{(|\alpha_c|, |\alpha_p|) \in [0, 1]^2} \log_2 \left(1 + \frac{(1 - |\alpha_c|^2)S}{1 + (1 - |\alpha_p|^2)I} \right) + \\ &\quad + \min \{ \log_2 (1 + C(1 - |\alpha_p|^2)), \log_2 (1 + S) \} . \\ &= \max_{|\alpha_p| \leq 1} \log_2 \left(1 + \frac{S}{1 + (1 - |\alpha_p|^2)I} \right) + \\ &\quad + \min \{ \log_2 (1 + C(1 - |\alpha_p|^2)), \log_2 (1 + S) \} . \end{aligned}$$

Since $\frac{S}{C} < 1$ (because $1 < \alpha_f$ in this regime) we choose $|\alpha_p|^2 = 1 - \frac{S}{C} \in [0, 1]$ (i.e., so that the two terms in the min are equal) and obtain

$$(R_p + R_c)^{(\text{IB})} \geq \log_2 \left(1 + \frac{S}{1 + I \frac{S}{C}} \right) + \log_2 (1 + S)$$

that implies

$$\begin{aligned} d(\alpha_i, \alpha_f) &\geq \lim_{S \rightarrow +\infty} \frac{(R_p + R_c)^{(\text{IB})}}{2 \log_2 (1 + S)} \\ &= \frac{\left[1 - [1 + \alpha_i - \alpha_f]^+ \right]^+ + 1}{2} \\ &= \begin{cases} 1 & \text{for } \alpha_f \geq \alpha_i + 1 \\ \frac{\alpha_f - \alpha_i + 1}{2} & \text{for } \alpha_i \leq \alpha_f < \alpha_i + 1 \\ \frac{1}{2} & \text{for } \alpha_f < \alpha_i \end{cases} . \end{aligned}$$

This shows the achievability of the DoF upper bound in the regime of interest $\alpha_f > 1, \alpha_i \leq 1$ and $\alpha_i \in [1, 2], \alpha_f \geq \alpha_i + 1$.
Outer Bound: when $\alpha_f \in [1, \alpha_i + 1]$ we use Tuninetti's upper bound with $I \leq S$

$$(R_p + R_c)^{(\text{OB})} \leq \log_2 \left(\frac{1 + C + S}{1 + I} \right) + \log_2 (1 + S + I + 2\sqrt{SI}) .$$

and when $\alpha_f > \alpha_i + 1$ we use the cut-set bound

$$(R_p + R_c)^{(\text{OB})} \leq 2 \log_2 (1 + S) .$$

Gap between OB and IB: For $\alpha_f \in [1, 1 + \alpha_i], \alpha_i \leq 1$

$$\begin{aligned} \text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\ &\leq \log_2 \left(\frac{1 + C + S}{1 + I} \right) + \log_2 (1 + S + I + 2\sqrt{SI}) + \\ &\quad - \log_2 (1 + S) - \log_2 \left(\frac{C + SI + SC}{C + SI} \right) \\ &\stackrel{\text{since } I \leq S}{\leq} \log_2 \left(\frac{1 + C + S}{C + SI + SC} \right) + \log_2 \left(\frac{C + SI}{1 + I} \right) + \log_2 \left(\frac{1 + 4S}{1 + S} \right) \\ &\stackrel{\text{since } C \leq IS}{\leq} \log_2 \left(\frac{1 + C + S}{C + SI + SC} \right) + \log_2 \left(\frac{2 SI}{1 + I} \right) + 2 \\ &\leq \log_2 \left(\frac{(1 + C + S)S}{C + SI + SC} \right) + 3 \\ &\stackrel{\text{since } I \geq 1}{\leq} \log_2 \left(\frac{(1 + C + S)S}{C + S + SC} \right) + 3 \\ &\stackrel{\text{using } S \leq C \leq +\infty}{\leq} \max \left\{ \log_2 \left(\frac{(1 + 2S)S}{S(2 + S)} \right), \log_2 \left(\frac{S}{1 + S} \right) \right\} + 3 \\ &\leq \max \{ \log_2(2), \log_2(1) \} + 3 = 4 \text{bits} . \end{aligned}$$

For $\alpha_f \geq \alpha_i + 1, \alpha_i \leq 1$,

$$\begin{aligned}
\text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\
&\leq 2 \log_2(1 + S) - \log_2(1 + S) - \log_2\left(\frac{C + SI + SC}{C + SI}\right) \\
&= \log_2\left(\frac{C + SI + SC + S^2I}{C + SI + SC}\right) \\
&= \log_2\left(1 + \frac{S^2I}{C + SI + SC}\right) \leq \log_2\left(1 + \frac{S^2I}{2SI + S^2I}\right) \leq 1 \text{bit}
\end{aligned}$$

considering that $C \geq SI$.

2) **Regime 2:** $\alpha_i \in [1, 2], \alpha_f \in [\alpha_i, 1 + \alpha_i]$.

Inner Bound: We use the achievable scheme in (19) from Appendix H-C, where we show that the

$$(R_p + R_c)^{(\text{IB})} \geq 2 \log_2(1 + S) - \log(2) - \log_2\left(\max\left\{1, \frac{(1 + S)(1 + \frac{C}{S} + S)}{(1 + I + S)(1 + \frac{C}{I})}\right\}\right)$$

which implies that the DoF satisfies

$$\begin{aligned}
d(\alpha_i, \alpha_f) &\geq \lim_{S \rightarrow +\infty} \frac{(R_p + R_c)^{(\text{IB})}}{2 \log_2(1 + S)} \\
&= 1 - \frac{1}{2} \left[1 + \max\{\alpha_f - 1, 1\} - \max\{\alpha_i, 1\} - \max\{\alpha_f - \alpha_i, 0\} \right]^+ \\
&\quad \alpha_i \in [1, 2], \alpha_f \in [\alpha_i, 1 + \alpha_i] \quad 1 - \frac{1}{2} \max\{0, 1 + 1 - \alpha_i - \alpha_f + \alpha_i\} \\
&= \frac{1}{2} \min\{2, \alpha_f\}.
\end{aligned}$$

This shows the achievability of the DoF upper bound.

Outer Bound: when $\alpha_f < 2$ we use Tuninetti's upper bound with $I \geq S$

$$(R_p + R_c)^{(\text{OB})} \leq \log_2\left(\frac{1 + C + I}{1 + I}\right) + \log_2(1 + S + I + 2\sqrt{SI})$$

and when $\alpha_f \geq 2$ we use the cut-set bound

$$(R_p + R_c)^{(\text{OB})} \leq 2 \log_2(1 + S).$$

Gap between OB and IB: we use the achievable sum-rate in (19). Since determining which function attains the minimum in (19) is quite involved analytically, we proceed as follows. The inner bound is given by $\min\{f_1, f_2\}$ and the outer bound by f_0 , for some functions $f_i, i \in [0 : 2]$; the gap hence is given by

$$\text{GAP} = f_0 - \min\{f_1, f_2\} = \max\{f_0 - f_1, f_0 - f_2\}.$$

We proceed to upper bound each term $f_0 - f_i, i \in [1 : 2]$ and we take the maximum of the obtained values as the final gap for this regime.

We start with the regime $I < C < S^2$ (i.e., subset of interest for $\alpha_f < 2$).

Let $f_1 = 2 \log_2(1 + S) - \log(2)$, then

$$\begin{aligned}
\text{GAP}_1 &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\
&\leq \log_2\left(\frac{1 + C + I}{1 + I} \cdot 1 + S + I + 2\sqrt{SI}\right) - 2 \log_2(1 + S) + 1 \\
&= 1 + \log_2\left(\frac{1 + S + I + 2\sqrt{SI}}{1 + I} \cdot \frac{1 + C + I}{(1 + S)^2}\right) \\
&\leq 1 + \log_2\left(\frac{1 + 4I}{1 + I} \cdot \frac{1 + 2S^2}{(1 + S)^2}\right) \\
&\leq 1 + \log_2(4 \cdot 2) = 4 \text{ bits}
\end{aligned}$$

since $I < C < S^2$ and $S < I$.

Now let $f_2 = \log_2 \left(\frac{1+I+S}{1+\frac{C}{S}+S} \right) + \log_2(1+S) + \log_2 \left(1 + \frac{C}{I} \right) - \log(2)$, then

$$\begin{aligned} \text{GAP}_2 &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\ &\leq \log_2 \left(\frac{1+C+I}{1+I} \right) + \log_2 \left(1 + S + I + 2\sqrt{SI} \right) + \\ &\quad - \log_2 \left(\frac{1+I+S}{1+\frac{C}{S}+S} \right) - \log_2(1+S) - \log_2 \left(1 + \frac{C}{I} \right) + \log(2) \\ &= 1 + \log_2 \left(\frac{1+C+I}{C+I} \cdot \frac{1+(\sqrt{S}+\sqrt{I})^2}{1+I+S} \cdot \frac{1+\frac{C}{S}+S}{1+S} \cdot \frac{I}{1+I} \right) \\ &\leq 1 + \log_2(1 \cdot 2 \cdot 2 \cdot 1) = 3 \text{ bits} \end{aligned}$$

since $S < I$ and $C < S^2$.

Thus

$$\text{GAP} = \max \{ \text{GAP}_1, \text{GAP}_2 \} \leq 4 \text{ bits.}$$

Now we turn to the regime $S^2 < C < SI$, $S < I < S^2$ (i.e., subset of interest for $\alpha_f > 2$).

With f_1 we have

$$\begin{aligned} \text{GAP}_3 &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\ &\leq 2 \log_2(1+S) - 2 \log_2(1+S) + 1 = 1 \text{ bit.} \end{aligned}$$

With f_2 we have

$$\begin{aligned} \text{GAP}_4 &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\ &\leq 2 \log_2(1+S) - \log_2 \left(\frac{1+I+S}{1+\frac{C}{S}+S} \right) - \log_2(1+S) - \log_2 \left(1 + \frac{C}{I} \right) + \log(2) \\ &= 1 + \log_2 \left(\frac{1+S}{S} \cdot \frac{I}{1+I+S} \cdot \frac{(1+S)S+C}{C+I} \right) \\ &\leq 1 + \log_2(2 \cdot 1 \cdot 2) = 3 \text{ bits.} \end{aligned}$$

since $\min\{C, S\} > 1$ and $S(S+1) \leq (S+1)^2 \leq (\sqrt{C}+1)^2$.

Thus

$$\text{GAP} = \max \{ \text{GAP}_3, \text{GAP}_4 \} \leq 3 \text{ bits.}$$

- 3) **Regime 3 (where the DoF is as for the non-cooperative channel):** $\alpha_f \leq \max\{1, \alpha_i\}$, $\alpha_i < 2$.

The analysis is similar to that of the strong interference regime with symmetric interfering links, where for achievable rate we use the sum-rate capacity of the non-cooperative S-channel [19, Theorem 2], and the gap is at most 2 bits.

- 4) **Regime 4 (where the DoF is as for the non-cooperative channel):** $\alpha_i \geq 2$. In this regime the sum-capacity is known exactly. In fact, for the non-cooperative channel in very strong interference the sum-rate capacity is $2 \log_2(1+S)$; therefore for the cooperative channel the sum-rate $2 \log_2(1+S)$ is achievable. From the cut-set upper bound for the cooperative S-channel we know that

$$\begin{aligned} R_p &\leq I(Y_p; X_p) \leq \log(1+S), \\ R_c &\leq I(Y_c; X_c | X_p) \leq \log(1+S) \end{aligned}$$

hence $2 \log_2(1+S)$ is also optimal for the cooperative S-channel in very strong interference and is achieved without the need of cooperation.

This demonstrates the achievability of the DoF in (12) to within a constant gap of 4 bits/s/Hz.

APPENDIX F ACHIEVABLE SCHEME 1

We consider [4, Th.IV.1], an achievable region for the interference channel with general source cooperation based on superposition coding only, with $T_1 = T_2 = U_1 = V_2 = \emptyset$, $V_1 = X_1$, $U_2 = X_2$, that is, $R_1 = R_{10c}$, $R_2 = R_{20n}$ (a cooperative-common message for user 1 and a non-cooperative-common message for user 2). Then the region

$$\begin{aligned} R_1 &\leq I(Y_2; X_1 | Q, X_2), \\ R_2 &\leq I(Y_4; X_2 | Q, X_1), \\ R_1 + R_2 &\leq \min_{j=3,4} I(Y_j; Q, X_1, X_2), \end{aligned}$$

is achievable for any input distribution $P_{Q,X_1,X_2} = P_Q P_{X_1|Q} P_{X_2|Q}$.

By identifying Node1 with the PTx (i.e., $X_p = X_1$), Node2 with the CTx (i.e., $X_c = X_2, Y_f = Y_2$), Node3 with the PRx (i.e., $Y_p = Y_3$) and Node4 with the CRx (i.e., $Y_c = Y_4$), the following sum-rate is achievable

$$(R_p + R_c)^{(\text{IB})} = \max_{P_Q P_{X_1|Q} P_{X_2|Q}} \min \left\{ I(Y_p; Q, X_c, X_p), I(Y_c; Q, X_c, X_p), \right. \\ \left. I(Y_f; X_p|X_c, Q) + I(Y_c; X_c|X_p, Q) \right\}.$$

In Gaussian noise, we choose Q, U_c, U_p be iid $\mathcal{N}(0, 1)$ and define

$$X_c = \sqrt{P_c}(\alpha_c Q + \beta_c U_c) : |\alpha_c|^2 + |\beta_c|^2 \leq 1, \\ X_p = \sqrt{P_p}(\alpha_p Q + \beta_p U_p) : |\alpha_p|^2 + |\beta_p|^2 \leq 1.$$

With this choice of inputs the channel outputs are

$$Y_j = h_{pj}X_p + h_{cj}X_c + Z_j \\ = (h_{pj}\sqrt{P_p}\alpha_p + h_{cj}\sqrt{P_c}\alpha_c)Q + h_{pj}\sqrt{P_p}\beta_p U_p + h_{cj}\sqrt{P_c}\beta_c U_c + Z_j, \quad j \in \{p, c, f\}.$$

By assuming $\angle h_{cc} - \angle h_{pc} = \angle h_{cp} - \angle h_{pp}$ (we assume so for analytical convenience so that we do not need to deal with the phase of the complex-valued channel gains; by lifting this assumption the rate expressions become more involved and therefore less amenable to analytical manipulations; the DoF result remains valid while the gap could be different deepening on the bounding technique), an achievable sum-rate is

$$(R_p + R_c)^{(\text{IB})} = \max_{(|\alpha_c|, |\alpha_p|) \in [0, 1]^2} \min \left\{ \right. \\ \log_2(1 + C(1 - |\alpha_p|^2)) + \log_2(1 + S(1 - |\alpha_c|^2)), \\ \left. \log_2(1 + S + I + 2\sqrt{SI} |\alpha_p||\alpha_c|) \right\} \\ \geq \begin{cases} \log_2(1 + C) + \log_2(1 + S) & \text{if } C(1 + S) \leq I, \\ \log_2(1 + S + I) & \text{if } C(1 + S) > I, \end{cases}$$

where the $(|\alpha_c|, |\alpha_p|) \in [0, 1]^2$ has been chosen as follows.

- Case 1: when

$$(1 + C)(1 + S) = \max_{(|\alpha_c|, |\alpha_p|) \in [0, 1]^2} \left\{ (1 + C(1 - |\alpha_p|^2)) (1 + S(1 - |\alpha_c|^2)) \right\} \\ < \min_{(|\alpha_c|, |\alpha_p|) \in [0, 1]^2} \left\{ (1 + S + I + 2\sqrt{SI} |\alpha_p||\alpha_c|) \right\} = 1 + S + I,$$

then $|\alpha_p| = |\alpha_c| = 0$ is optimal; this case occurs when $C(1 + S) < I$ in which case the min-max expression is equal to $\log(1 + C) + \log(1 + S)$.

- Case 2: when

$$1 + (\sqrt{S} + \sqrt{I})^2 = \max_{(|\alpha_c|, |\alpha_p|) \in [0, 1]^2} \left\{ (1 + S + I + 2\sqrt{SI} |\alpha_p||\alpha_c|) \right\} \\ < \min_{(|\alpha_c|, |\alpha_p|) \in [0, 1]^2} \left\{ (1 + C(1 - |\alpha_p|^2)) (1 + S(1 - |\alpha_c|^2)) \right\} = 1,$$

then $|\alpha_p| = |\alpha_c| = 1$ is optimal; this case never occurs.

- Case 3: for the parameter values not covered by Case 1 and Case 2, that is $C(1 + S) \geq I$, the optimal is given by equating the two expressions in the min. By doing so we obtain

$$a := \frac{S/C |\alpha_c|^2}{1 + S(1 - |\alpha_c|^2)}, \\ b := \frac{I/C}{1 + S(1 - |\alpha_c|^2)}, \\ |\alpha_p| = \sqrt{(1 - a)(1 - b)} - \sqrt{ab},$$

where the above solution for $|\alpha_p|$ is in the range $[0, 1]$ if and only if $a + b \leq 1$ (in the following we make sure that the solution we pick satisfies $a + b \leq 1$ so that we do not need to care here about the case $a + b > 1$). Hence, for $C(1 + S) \geq I$ we have

$$(R_c + R_p)^{(\text{IB})} = \max_{|\alpha_c| \in [0, 1]: a+b \leq 1} \log_2 \left(1 + S(1 - b|\alpha_c|^2) + I(1 - a) + 2|\alpha_c|\sqrt{SI(1 - a)(1 - b)} \right).$$

Instead of solving analytically for the optimal value of $|\alpha_c|$, which does not seem to lead to a closed-form expression, we choose the possibly suboptimal value $|\alpha_c| = 0$ to obtain

$$(R_c + R_p)^{(\text{IB})} \geq \log_2(1 + S + I).$$

Note that for $|\alpha_c| = 0$

$$a + b|_{|\alpha_c|=0} = 0 + \frac{I/C}{1+S} \leq 1$$

by definition of Case 3, therefore $|\alpha_c| = 0$ is a valid solution for Case 3.

To summarize, with this scheme the following sum-rate is achievable

$$(R_p + R_c)^{(\text{IB})} \geq \begin{cases} \log_2(1 + C) + \log_2(1 + S) & \text{if } C(1 + S) \leq I, \\ \log_2(1 + S + I) & \text{if } C(1 + S) > I, \end{cases} \quad (13)$$

APPENDIX G ACHIEVABLE SCHEME 2

We use [4, Theorem 4.1], an achievable region for the interference channel with general source cooperation based on superposition coding only, with $U_1 = V_2 = \emptyset$, i.e., then $R_1 = R_{10c} + R_{11n}$, $R_2 = R_{20n} + R_{22n}$ (both users have common and private messages; only the common message of user 1 is sent cooperatively). With this choice, one can easily see that

$$0 \leq [4, \text{eq.}(6a)], \quad [4, \text{eq.}(6b)] = [4, \text{eq.}(6d)] \leq [4, \text{eq.}(6c)] = [4, \text{eq.}(6e)] \leq [4, \text{eq.}(6f)],$$

and

$$0 = [4, \text{eq.}(7a)] \leq [4, \text{eq.}(7b)] = [4, \text{eq.}(7c)] \leq [4, \text{eq.}(7d)] = [4, \text{eq.}(7e)] \leq [4, \text{eq.}(7f)],$$

which imply that in [4, Theorem 4.1] the following bounds are redundant

$$[4, \text{eq.}(8c)], [4, \text{eq.}(8h)], [4, \text{eq.}(8i)], [4, \text{eq.}(8j)], [4, \text{eq.}(8k)] \text{ and } [4, \text{eq.}(8m)].$$

By applying Fourier-Motzkin elimination on the resulting region in [4, Theorem 4.1] so as to obtain an achievable sum-rate we get

$$(R_p + R_c)^{(\text{IB})} = \max \min \{ \min \{ [4, \text{eq.}(8a)], [4, \text{eq.}(8b)] \} + [4, \text{eq.}(8d)] \quad (14a)$$

$$[4, \text{eq.}(8e)], [4, \text{eq.}(8f)], [4, \text{eq.}(8g)], \quad (14b)$$

$$\frac{\min \{ [4, \text{eq.}(8a)], [4, \text{eq.}(8b)] \} + [4, \text{eq.}(8\ell)]}{2} \}, \quad (14c)$$

where the maximization is over all distributions $P_{Q,V_1,T_1,X_1,U_2,T_2,X_2} = P_Q P_{V_1,T_1,X_1|Q} P_{U_2,T_2,X_2|Q}$.

By identifying Node1 with the PTx (i.e., $X_p = X_1$), Node2 with the CTx (i.e., $X_c = X_2, Y_f = Y_2$), Node3 with the PRx (i.e., $Y_p = Y_3$) and Node4 with the CRx (i.e., $Y_c = Y_4$), the following sum-rate is achievable. In Gaussian noise, we choose $Q = \emptyset$, V_1, T_1, U_2, T_2 be iid $\mathcal{N}(0, 1)$ and let

$$X_c = \sqrt{P_c}(\alpha_c U_2 + \beta_c T_2) : |\alpha_c|^2 + |\beta_c|^2 \leq 1,$$

$$X_p = \sqrt{P_p}(\alpha_p V_1 + \beta_p T_1) : |\alpha_p|^2 + |\beta_p|^2 \leq 1.$$

With this choice of inputs the channel outputs are

$$\begin{aligned} Y_j &= h_{pj} X_p + h_{cj} X_c + Z_j \\ &= h_{pj} \sqrt{P_p} \alpha_p V_1 + h_{cj} \sqrt{P_c} \alpha_c U_2 + h_{pj} \sqrt{P_p} \beta_p T_1 + h_{cj} \sqrt{P_c} \beta_c T_2 + Z_j, \quad j \in \{f, p, c\}. \end{aligned}$$

Under these conditions we obtain

$$\begin{aligned}
[4, eq.(6a)] &:= I(Y_f; V_1 | U_2, T_2, X_p) = \log_2 \left(\frac{1+C}{1+(1-|\alpha_p|^2)C} \right) \\
[4, eq.(6b)] = [4, eq.(6d)] &:= I(Y_p; T_1 | V_1, U_2) = \log_2 \left(1 + \frac{(1-|\alpha_p|^2)S}{1+(1-|\alpha_c|^2)I_p} \right) \\
[4, eq.(6c)] = [4, eq.(6e)] &:= I(Y_p; T_1, U_2 | V_1) = \log_2 \left(\frac{1+I_p+(1-|\alpha_p|^2)S}{1+(1-|\alpha_c|^2)I_p} \right) \\
[4, eq.(6f)] &:= I(Y_p; T_1, U_2, V_1) = \log_2 \left(\frac{1+S+I_p}{1+(1-|\alpha_c|^2)I_p} \right) \\
[4, eq.(7b)] = [4, eq.(7c)] &:= I(Y_c; T_2 | V_1, U_2) = \log_2 \left(1 + \frac{(1-|\alpha_c|^2)S}{1+(1-|\alpha_p|^2)I_c} \right) \\
[4, eq.(7d)] = [4, eq.(7e)] &:= I(Y_c; T_2, U_2 | V_1) = \log_2 \left(1 + \frac{S}{1+(1-|\alpha_p|^2)I_c} \right) \\
[4, eq.(7f)] &:= I(Y_c; T_2, U_2, V_1) = \log_2 \left(\frac{1+S+I_c}{1+(1-|\alpha_p|^2)I_c} \right).
\end{aligned}$$

where, for later use, we indicated the interference at the PRx by $I_p = |h_{cp}|^2 P_c$ and the interference at the CRx by $I_c = |h_{pc}|^2 P_p$.

For the interference-symmetric case, i.e., $I_c = I_p = I$, inspired by the scheme of Etkin-Tse-Wang for the classical interference channel in weak interference, we set $(1-|\alpha_c|^2)I = (1-|\alpha_p|^2)I = 1$, to get

$$\begin{aligned}
[4, eq.(6a)] &:= I(Y_f; V_1 | U_2, T_2, X_p) = \log_2 \left(\frac{1+C}{1+\frac{C}{I}} \right) \\
[4, eq.(6b)] = [4, eq.(6d)] &:= I(Y_p; T_1 | V_1, U_2) = \log_2 \left(1 + \frac{\frac{S}{I}}{2} \right) \\
[4, eq.(6c)] = [4, eq.(6e)] &:= I(Y_p; T_1, U_2 | V_1) = \log_2 \left(\frac{\frac{S}{I} + I + 1}{2} \right) \\
[4, eq.(6f)] &:= I(Y_p; T_1, U_2, V_1) = \log_2 \left(\frac{S + I + 1}{2} \right) \\
[4, eq.(7b)] = [4, eq.(7c)] &:= I(Y_c; T_2 | V_1, U_2) = \log_2 \left(1 + \frac{\frac{S}{I}}{2} \right) \\
[4, eq.(7d)] = [4, eq.(7e)] &:= I(Y_c; T_2, U_2 | V_1) = \log_2 \left(1 + \frac{S}{2} \right) \\
[4, eq.(7f)] &:= I(Y_c; T_2, U_2, V_1) = \log_2 \left(\frac{S + I + 1}{2} \right).
\end{aligned}$$

Now we evaluate achievable sum-rate using (14), where

$$\begin{aligned}
[4, eq.(8a)] &= [4, eq.(6f)] = \log_2 \left(\frac{S+I+1}{2} \right) \\
[4, eq.(8b)] &= [4, eq.(6a)] + [4, eq.(6d)] = \log_2 \left(\frac{1+C}{1+\frac{C}{I}} \right) + \log_2 \left(1 + \frac{\frac{S}{I}}{2} \right) \\
[4, eq.(8d)] &= [4, eq.(7d)] = \log_2 \left(1 + \frac{S}{2} \right) \\
[4, eq.(8e)] &= [4, eq.(6f)] + [4, eq.(7b)] = \log_2 \left(\frac{S+I+1}{2} \right) + \log_2 \left(1 + \frac{\frac{S}{I}}{2} \right) \\
[4, eq.(8f)] &= [4, eq.(7f)] + [4, eq.(6b)] = \log_2 \left(1 + \frac{\frac{S}{I}}{2} \right) + \log_2 \left(\frac{S+I+1}{2} \right) \\
[4, eq.(8g)] &= [4, eq.(6a)] + [4, eq.(6e)] + [4, eq.(7b)] \\
&= \log_2 \left(\frac{1+C}{1+\frac{C}{I}} \right) + \log_2 \left(\frac{\frac{S}{I}+I+1}{2} \right) + \log_2 \left(1 + \frac{\frac{S}{I}}{2} \right) \\
[4, eq.(8l)] &= [4, eq.(6c)] + [4, eq.(7b)] + [4, eq.(7f)] \\
&= \log_2 \left(\frac{\frac{S}{I}+I+1}{2} \right) + \log_2 \left(1 + \frac{\frac{S}{I}}{2} \right) + \log_2 \left(\frac{S+I+1}{2} \right).
\end{aligned}$$

We next show that the sum-rate in (14) is equal to the term in (14b). In order to show that the term in (14a) is redundant, consider the following facts:

- $[4, eq.(8a)] + [4, eq.(8d)]$ is always greater than $[4, eq.(8e)]$ because

$$S \geq S/I,$$

since we assume $I \geq 1$.

- $[4, eq.(8b)] + [4, eq.(8d)]$ is always greater than $[4, eq.(8g)]$ since

$$2I + SI \geq S + I^2 + I \iff S \geq I,$$

which is always satisfied since we are in the weak interference regime.

In order to show that the term in (14c) is redundant, consider the following facts:

- the bound $\frac{[4, eq.(8a)] + [4, eq.(8l)]}{2}$ is always bigger than $[4, eq.(8e)]$ and it is therefore redundant.
- the bound $\frac{[4, eq.(8b)] + [4, eq.(8l)]}{2}$ is equal to $\frac{[4, eq.(8f)] + [4, eq.(8g)]}{2}$ and it is therefore redundant.

Therefore we conclude that in the weak interference regime the sum-rate in (14) is equal to (14b) and, since $[4, eq.(8f)]$ is equal to $[4, eq.(8e)]$, is given by

$$\begin{aligned}
(R_p + R_c)^{(IB)} &= \min \{ [4, eq.(8e)], [4, eq.(8g)] \} \\
&= \min \left\{ \log_2 \left(\frac{2I+S}{2I} \right) + \log_2 \left(\frac{S+I+1}{2} \right), \right. \\
&\quad \left. \log_2 \left(\frac{2I+S}{2I} \right) + \log_2 \left(\frac{1+C}{I+C} \right) + \log_2 \left(\frac{S+I^2+I}{2} \right) \right\}.
\end{aligned}$$

For future use we note that the second term in the above min is the smallest term if

$$(S+I+1)(I+C) \geq S + I^2 + I + SC + CI^2 + CI \iff S \geq C(I+1).$$

To summarize, with this scheme the following sum-rate is achievable

$$(R_p + R_c)^{(IB)} \geq \min \left\{ \log_2 \left(\frac{2I+S}{2I} \right) + \log_2 \left(\frac{S+I+1}{2} \right), \right. \quad (15a)$$

$$\left. \log_2 \left(\frac{2I+S}{2I} \right) + \log_2 \left(\frac{1+C}{I+C} \right) + \log_2 \left(\frac{S+I^2+I}{2} \right) \right\}. \quad (15b)$$

APPENDIX H

DPC-BASED ACHIEVABLE SCHEME

We use [4, Theorem V.1], an achievable region for the interference channel with general source cooperation based on Gelfand-Pinsker binning and superposition, with $U_1 = T_1 = S_2 = V_2 = U_2 = Z_2 = \emptyset$, i.e., then $R_1 = R_{10c} + R_{11c}$, $R_2 = R_{22n}$. This scheme corresponds to a cooperative common message (carried by (Q, V_1) at rate R_{10c}) for user 1, a cooperative private message (carried by (S_1, Z_1) at rate R_{11c}) for user 1, and a non-cooperative private message (carried by T_2 at rate R_{22n}) for user 2. Here S_1 , reps. Q , represents the “past cooperative private message”, reps. “past cooperative common message”, and Z_1 , reps. V_1 , the “new cooperative private message”, reps. “new cooperative common message”, in a block Markov encoding scheme. The channel inputs are function of the auxiliary random variables, with X_1 is a function of (Q, S_1, V_1, Z_1) and X_2 a function of (Q, T_2, S_1) . [4, Theorem V.1] involves several binning steps to allow for a large set of possible input distributions. Here, in order to simplify the scheme, we do not bin (V_1, Z_1) against S_1 ; the only binning step is for T_2 against S_1 . With this we have the following achievable region.

The set of possible input distributions is

$$P_{Q,S_1,V_1,Z_1,X_1,T_2,X_2} = P_Q P_{S_1|Q} P_{V_1|Q} P_{Z_1|S_1,V_1,Q} P_{T_2|S_1,Q} P_{X_1|S_1,V_1,Z_1,Q} P_{X_2|S_1,T_2,Q}. \quad (16)$$

Regarding encoding. The T_2 -codebook is generated independently of the S_1 -codebook given the Q -codebook; by binning, the transmitted codewords appears to follow a joint distribution $P_{S_1,T_2|Q}$; for this to be possible, we must generate several T_2 -sequences for each message of user 2 to be able to find one to send with the correct joint distribution; this entails a rate penalty equal to $I(S_1; T_2|Q)$ for the rate of user 2.

Regarding decoding. There are three decoding nodes in the network and therefore three groups of rate constraints. These are:

- Node 2/CTx jointly decodes (V_1, Z_1) from its channel output with knowledge of (Q, S_1, T_2, X_2) . Successful decoding is possible if (see [4, eq(21)])

$$R_{10c} + R_{11c} \leq I(Z_1, V_1; Y_2 | T_2, X_2, S_1, Q), \quad (17a)$$

$$R_{11c} \leq I(Z_1; Y_2 | T_2, X_2, S_1, V_1, Q). \quad (17b)$$

- Node 3/PRx jointly decodes (Q, S_1) from its channel output with knowledge of the message index of (V_1, Z_1) ; Of the 28 error events listed in [4, Table 1] only the events number 0 and 12 matter (all other events are either equal to these two or are not an error given that many random variables defining the scheme have been set to zero); this implies that successful decoding is possible if

$$R_{10c} + R_{11c} \leq I(Y_3; Q, V_1, S_1, Z_1), \quad (17c)$$

$$R_{11c} \leq I(Y_3; S_1, Z_1 | V_1, Q). \quad (17d)$$

- Node 4/CRx jointly decodes (Q, T_2) from its channel output, with knowledge of the message index of V_1 , by treating Z_1 as noise (recall that T_2 has been precoded/binning against S_1). Of the 28 error events listed in [4, Table 1] with the role of the source-destination pairs swapped only the events number 0 and 1 matter (all other events are either equal to these two or are not an error given that many random variables defining the scheme have been set to zero); this implies that successful decoding is possible if

$$R_{10c} + R_{22n} \leq I(Y_4; T_2, V_1, Q) - I(T_2; S_1 | Q), \quad (17e)$$

$$R_{22n} \leq I(Y_4; T_2 | V_1, Q) - I(T_2; S_1 | Q). \quad (17f)$$

By identifying Node1 with the PTx (i.e., $X_p = X_1$), Node2 with the CTx (i.e., $X_c = X_2, Y_f = Y_2$), Node3 with the PRx (i.e., $Y_p = Y_3$) and Node4 with the CRx (i.e., $Y_c = Y_4$), the following region is achievable

$$(R_p + R_c)^{(\text{IB})} = \max \min \left\{ \begin{aligned} & \min \{ I(Z_1, V_1; Y_f | T_2, X_c, S_1, Q), I(Y_p; V_1, Z_1, S_1, Q) \} + I(Y_c; T_2 | V_1, Q) - I(T_2; S_1 | Q), \\ & \min \{ I(Z_1; Y_f | S_1, T_2, X_c, V_1, Q), I(Y_p; S_1, Z_1 | V_1, Q) \} + I(Y_c; T_2, V_1, Q) - I(T_2; S_1 | Q) \}, \end{aligned} \right.$$

where the maximization is over all distributions in (16).

A. Achievable scheme 3

By setting $Q = V_1 = \emptyset, Z_1 = X_p$ in the DPC-based scheme the following sum-rate is achievable.

$$(R_p + R_c)^{(\text{IB})} = \max_{P_{S_1} P_{X_p|S_1} P_{T_2, X_c|S_1}} \min \{ I(X_p; Y_f | S_1, T_2, X_c), I(S_1, X_p; Y_p) \} + I(T_2; Y_c) - I(T_2; S_1).$$

In Gaussian noise, we choose S_1, U_c, U_p be iid $\mathcal{N}(0, 1)$ and define

$$\begin{aligned} X_c &= \sqrt{P_c}(\alpha_c S_1 + \beta_c U_c) : |\alpha_c|^2 + |\beta_c|^2 \leq 1, \\ X_p &= \sqrt{P_p}(\alpha_p S_1 + \beta_p U_p) : |\alpha_p|^2 + |\beta_p|^2 \leq 1, \\ T_2 &= U_c + \lambda_{\text{Costa1}} S_1, \end{aligned}$$

and we choose λ_{Costa1} as in [3] so as to “pre-cancel” S_1 from Y_c in decoding U_c , i.e., so as to have

$$I(T_2; Y_c) - I(T_2; S_1) = I(T_2; Y_c | S_1).$$

With this choice of inputs the channel outputs are

$$\begin{aligned} Y_j &= h_{pj} X_p + h_{cj} X_c + Z_j \\ &= (h_{pj} \sqrt{P_p} \alpha_p + h_{cj} \sqrt{P_c} \alpha_c) S_1 + h_{pj} \sqrt{P_p} \beta_p U_p + h_{cj} \sqrt{P_c} \beta_c U_c + Z_j, \quad j \in \{p, c, f\}. \end{aligned}$$

By assuming $\angle h_{cc} - \angle h_{pc} = \angle h_{cp} - \angle h_{pp}$ (we assume so for analytical convenience so that we do not need to deal with the phase of the complex-valued channel gains; by lifting this assumption the rate expressions become more involved and therefore less amenable to analytical manipulations; the DoF result remains valid while the gap could be different deepening on the bounding technique), an achievable sum-rate is

$$\begin{aligned} (R_p + R_c)^{(\text{IB})} &= \max_{(|\alpha_c|, |\alpha_p|) \in [0, 1]^2} \log_2 \left(1 + \frac{(1 - |\alpha_c|^2)S}{1 + (1 - |\alpha_p|^2)I_c} \right) + \\ &+ \min \left\{ \log_2 (1 + C(1 - |\alpha_p|^2)), \log_2 \left(1 + \frac{S + |\alpha_c|^2 I_p + 2|\alpha_c| |\alpha_p| \sqrt{S I_p}}{1 + (1 - |\alpha_c|^2)I_p} \right) \right\}, \end{aligned}$$

where, for later use, we indicated the interference at the PRx by I_p and the interference at the CRx by I_c .

For the interference-symmetric case, i.e., $I_c = I_p = I$, instead of solving analytically the optimization above, which does not seem to lead to a closed-form expression, we choose to set the two functions in the min to be equal, which is a possibly a suboptimal solution. By doing so, we formally have to solve the same problem as in the “very strong interference and strong cooperation” regime, albeit with the role of S and I swapped. We obtain

$$\begin{aligned} a &:= \frac{S/C}{1 + I(1 - |\alpha_c|^2)}, \\ b &:= \frac{I/C |\alpha_c|^2}{1 + I(1 - |\alpha_c|^2)}, \\ |\alpha_p| &= \sqrt{(1 - a)(1 - b) - \sqrt{ab}}, \end{aligned}$$

and such that $a + b \leq 1$ (in the following we make sure that the solution we pick satisfies $a + b \leq 1$ so that we do not need to care here about the case $a + b > 1$) then

$$\begin{aligned} (R_p + R_c)^{(\text{IB})} &\geq \max_{|\alpha_c| \in [0, 1]: a+b \leq 1} \log_2 \left(\frac{1 + S(1 - b) + I(1 - a|\alpha_c|^2) + 2|\alpha_c| \sqrt{S I} \sqrt{(1 - a)(1 - b)}}{1 + (1 - |\alpha_c|^2)I} \right) \\ &+ \log_2 \left(1 + \frac{(1 - |\alpha_c|^2)S}{1 + (1 - (\sqrt{(1 - a)(1 - b)} - \sqrt{ab})^2)I} \right). \end{aligned}$$

We can further set $|\alpha_c| = 0$, which is possible if

$$a + b|_{|\alpha_c|=0} = \frac{S/C}{1 + I} \leq 1.$$

In the regime $\frac{S/C}{1+I} > 1$, we choose the possibly suboptimal solution, such as for example $|\alpha_c| = |\alpha_p| = 0$.

To summarize, with this scheme the following sum-rate is achievable

$$(R_p + R_c)^{(\text{IB})} \geq \log_2 \left(1 + \frac{S + I}{1 + I} \right) + \log_2 \left(\frac{1 + S}{1 + \frac{S/C}{1+I} I} \right) \quad \text{for } C \geq \frac{S}{1 + I}. \quad (18a)$$

$$(R_p + R_c)^{(\text{IB})} \geq \log_2 \left(1 + \frac{S}{1 + I} \right) + \log_2 (1 + C), \quad \text{for } C < \frac{S}{1 + I}. \quad (18b)$$

B. Gap for the regime $\alpha_i < 1$ and $\alpha_f > 1$ for the symmetric Gaussian CCIC

With the DPC-based achievable scheme in Appendix H-A an achievable sum-rate is given by (18a) which we now use to derive a smaller gap in the regime $I < S$ and $C > S$. The achievable sum-rate implies

$$\begin{aligned} d(\alpha_i, \alpha_f) &\geq \lim_{S \rightarrow \infty} \frac{(R_p + R_c)^{(\text{IB})}}{2 \log_2(1 + S)} \\ &= \lim_{S \rightarrow \infty} \frac{\log_2 \left(1 + \frac{S+I}{1+I} \right) + \log_2 \left(\frac{1+S}{1+\frac{S/C}{1+I}I} \right)}{2 \log_2(1 + S)} \\ &= \frac{1}{2} ([\max\{1, \alpha_i\} - \alpha_i]^+ + 1 - [1 - \alpha_f]^+) \\ &\stackrel{\text{since } \alpha_i < 1, \alpha_f > 1}{=} \frac{2 - \alpha_i}{2}. \end{aligned}$$

This shows the achievability of the DoF upper bound.

By using the upper bound in (5) under the condition $S \geq I$ we obtain the following gap

$$\begin{aligned} \text{GAP} &= (R_p + R_c)^{(\text{OB})} - (R_p + R_c)^{(\text{IB})} \\ &\leq \log_2 \left(\frac{1+S}{1+I} \right) + \log_2 (1 + S + I + 2\sqrt{SI}) + \\ &\quad - \log_2 \left(1 + \frac{S+I}{1+I} \right) - \log_2 \left(\frac{1+S}{1+\frac{S/C}{1+I}I} \right) \\ &\leq \log_2 \left(1 + \frac{2\sqrt{SI}}{1+S+2I} \right) + \log_2 \left(1 + \frac{S}{C} \right) \\ &\leq 2 \log_2(1 + 1) = 2 \text{ bits.} \end{aligned}$$

using $0 \leq I$ and $S \leq C$.

This example shows that an achievable scheme more complex than simple superposition coding, like the DPC-based one, can achieve a smaller gap.

C. Achievable scheme 4

In Gaussian noise, we let $Q = \emptyset$ and let V_1, S_1, Z_1, L_2 be iid $\mathcal{N}(0, 1)$ in the DPC-based scheme and define

$$\begin{aligned} X_p &= \sqrt{P_p}(aS_1 + bZ_1 + cV_1) : |a|^2 + |b|^2 + |c|^2 \leq 1, \\ X_c &= \sqrt{P_c}L_2, \\ T_2 &= L_2 + \lambda_{\text{Costa2}}S_1, \end{aligned}$$

where λ_{Costa2} is chosen as in [3] so as to “pre-cancel” S_1 from Y_c in decoding L_2 after having decoded V_1 while treating Z_1 as noise, that is

$$I(Y_c; T_2 | V_1) - I(T_2; S_1) = I(Y_c; T_2 | V_1, S_1).$$

With this the received signals for the S-channel are

$$\begin{aligned} Y_f &= \sqrt{C}(aS_1 + bZ_1 + cV_1) + Z_f, \\ Y_p &= \sqrt{S}(aS_1 + bZ_1 + cV_1) + Z_p, \\ Y_c &= \sqrt{S}L_2 + \sqrt{I}(aS_1 + bZ_1 + cV_1) + Z_c. \end{aligned}$$

and we obtain

$$\begin{aligned} R_{10c} + R_{11c} &\leq \log_2 (1 + |b|^2 C + |c|^2 C) = \log_2 (1 + (1 - |a|^2)C) \\ R_{11c} &\leq \log_2 (1 + |b|^2 C) \\ R_{10c} + R_{11c} &\leq \log_2 (1 + S) \\ R_{11c} &\leq \log_2 (1 + |a|^2 S + |b|^2 S) \\ R_{10c} + R_{22n} &\leq \log_2 \left(\frac{1 + I + S}{1 + |a|^2 I + |b|^2 I + S} \right) + \log_2 \left(1 + \frac{S}{1 + |b|^2 I} \right) \\ R_{22n} &\leq \log_2 \left(1 + \frac{S}{1 + |b|^2 I} \right) \end{aligned}$$

where $(|a|^2, |b|^2) \in [0, 1]^2 : |a|^2 + |b|^2 \leq 1$.

D. An achievable scheme for the S-channel in the regime $\alpha_i \in [1, 2], \alpha_f \in [\alpha_i, 1 + \alpha_i]$

Next we aim to obtain an achievable scheme for the S-channel in the regime $\alpha_i \in [1, 2], \alpha_f \in [\alpha_i, 1 + \alpha_i] \iff I \in [S, S^2], C \in [I, SI]$ that achieves the optimal DoF = $\min\{2, \alpha_f\}/2$. To this end we choose

$$|b|^2 I = 1, \quad |a|^2 S + |b|^2 S = |b|^2 C,$$

which is a feasible solution if $S \leq C \leq SI$, which is precisely the definition of the regime of interest. With this choice we obtain

$$\begin{aligned} R_{10c} + R_{11c} &\leq \log_2 \left(1 + C + \frac{C}{I} - \frac{C^2}{SI} \right) \\ R_{11c} &\leq \log_2 \left(1 + \frac{C}{I} \right) \\ R_{10c} + R_{11c} &\leq \log_2 (1 + S) \\ R_{11c} &\leq \log_2 \left(1 + \frac{C}{I} \right) \\ R_{10c} + R_{22n} &\leq \log_2 \left(\frac{1 + I + S}{1 + \frac{C}{S} + S} \right) + \log_2 \left(1 + \frac{S}{2} \right) \\ R_{22n} &\leq \log_2 \left(1 + \frac{S}{2} \right) \end{aligned}$$

Next, since $S \leq C \leq SI$ also implies $S \leq C + \frac{C}{I} - \frac{C^2}{SI}$, we have

$$\begin{aligned} &(R_p + R_c)^{(\text{IB})} \\ &\geq \min \left\{ \log_2 \left(1 + \frac{S}{2} \right) + \log_2 (1 + S), \right. \\ &\quad \left. \log_2 \left(\frac{1 + I + S}{1 + \frac{C}{S} + S} \right) + \log_2 \left(1 + \frac{S}{2} \right) + \log_2 \left(1 + \frac{C}{I} \right) \right\} \\ &\geq 2 \log_2 (1 + S) - \log(2) - \log_2 \left(\max \left\{ 1, \frac{(1 + S)(1 + \frac{C}{S} + S)}{(1 + I + S)(1 + \frac{C}{I})} \right\} \right) \end{aligned} \tag{19}$$

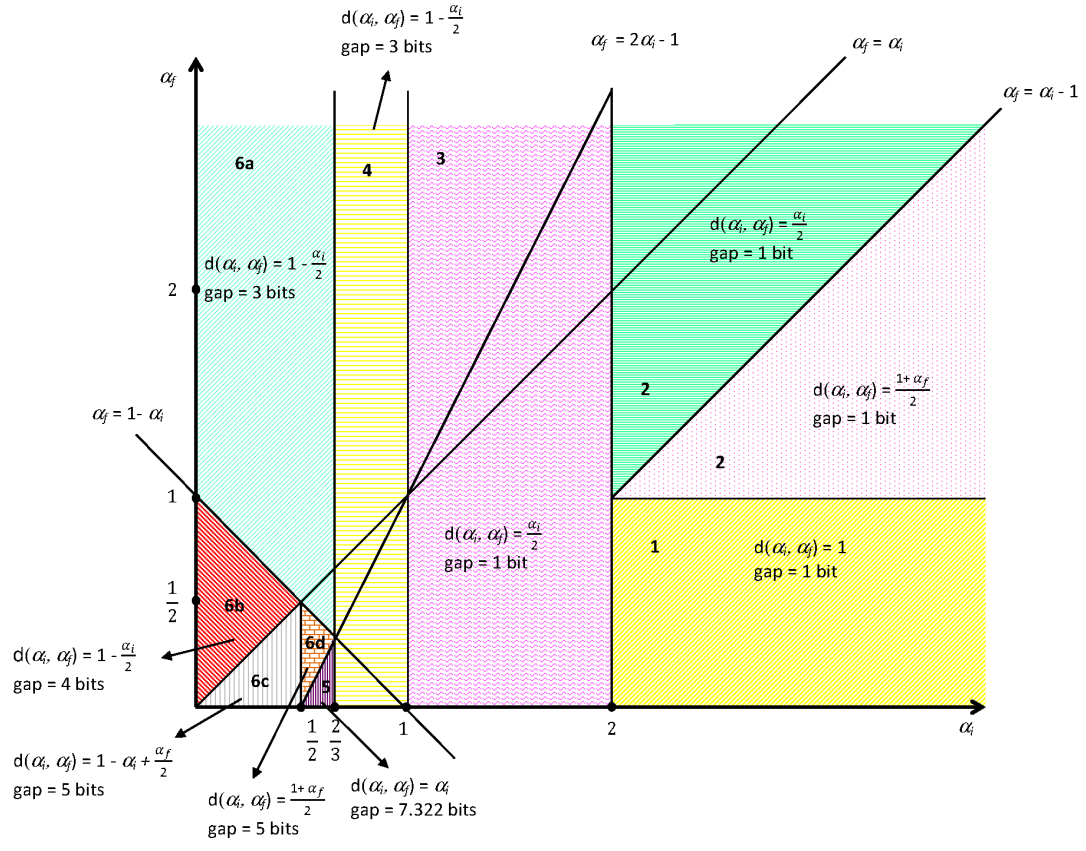


Fig. 3. Optimal DoF and constant gap for the symmetric G-CCIC in the different regimes of (α_i, α_f) .

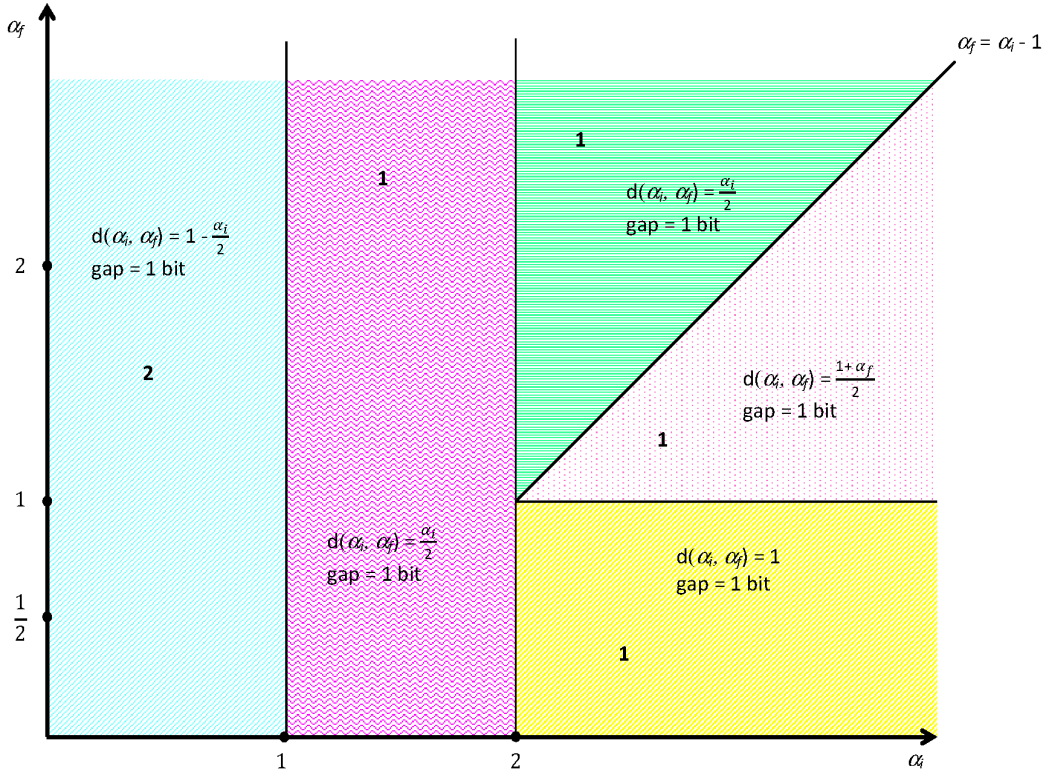


Fig. 4. Optimal DoF and constant gap for the Z-channel in the different regimes of (α_i, α_f) .

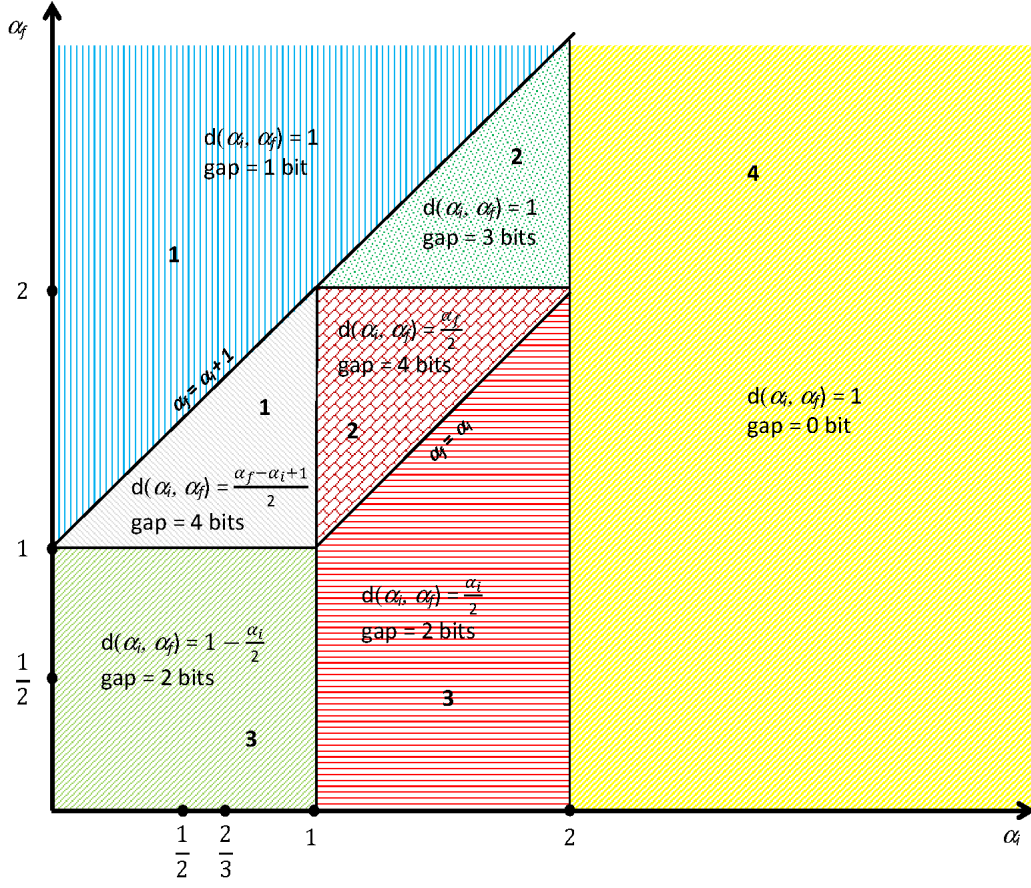


Fig. 5. Optimal DoF and constant gap for the S-channel in the different regimes of (α_i, α_f) .